Declining Search Frictions, Unemployment and Growth*

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Abstract

Over the last century, unemployment, vacancy, job-finding and job-loss rates as well as the Beveridge curve have no trend. Yet, the last century has seen the development and diffusion of many information technologies—such as telephones, fax machines, computers, the Internet—which presumably have increased the efficiency of search in the labor market. We explain this phenomenon using a textbook search-theoretic model of the labor market. We show that there exists an equilibrium in which unemployment, vacancies, job-finding and job-loss rates are constant while the search technology improves over time if and only if firm-worker matches are heterogeneous in quality, the distribution of match qualities is Pareto, and the quality of a match is observed before the start of the employment relationship. Under these conditions, improvements in search lead to an increase in the rate at which workers meet firms and to a proportional decline in the probability that the quality of a firm-worker match is acceptable leading to a constant job-finding rate, unemployment, etc... Interestingly, under the same conditions, unemployment, vacancies, job-finding and job-loss rates are independent of the size of the labor market even in the presence of increasing returns to scale in search. While declining search frictions do not lower unemployment, they contribute to growth. The magnitude of the contribution depends on the thickness of the tail of the Pareto distribution. We present a simple strategy to measure the decline in search frictions and its contribution to growth. A rudimentary implementation of this strategy suggests that the decline in search frictions has been substantial, it has been caused by both improvements in the search technology and increasing returns to scale in the search process, and it has had a non-negligible impact on growth.

JEL Codes: E24, O40, R11.

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1 Introduction

The US Beveridge curve has been surprisingly stable ever since data on unemployment and vacancies have begun being collected in the 1920s.[1] As illustrated in Figure 1, the US Beveridge curve has been stable during the five decades going from 1927 to 1976, it has shifted out during the 1977-1986 decade, shifted back in during the 1987-1996 decade, and it has again shifted out and back in during the ten years from 2007 to 2016. While much research has been devoted to the cyclical, counter-clockwise shifts of the Beveridge curve (see, e.g. Kaplan and Menzio 2016, Gavazza, Mongey and Violante 2018 and Sniekers 2018), what we find truly remarkable in this figure is the lack of any secular trend in the Beveridge curve. Indeed, in the aftermath of the Great Recession, the Beveridge curve is exactly where it was in the early 1950s.

There also have been no secular shifts along the Beveridge curve. As illustrated in Figure 2, the unemployment rate is quite volatile at the business cycle frequency, but it displays no clear secular trend. The vacancy rate is also very volatile at the business cycle frequency, although in the opposite direction, and does not feature any recognizable secular trend. As the unemployment and the vacancy rates have no trend, neither does the tightness of the labor market, which is defined as the vacancy-to-unemployment ratio.

The standard theory of unemployment, vacancies, and the Beveridge curve has been developed by Diamond (1982), Mortensen (1982) and Pissarides (2000). The theory is based on the view that unemployment and vacancies coexist because searching the labor market for a trading partner is a time-consuming activity. The relationship between unemployment and vacancies is downward sloping because the vacancy-to-unemployment ratio increases the speed at which unemployed workers find jobs and, hence, lowers unemployment. Formally, the theory states that the Beveridge curve is given by

$$u_t = \frac{\delta_t}{\delta_t + A_t p(v_t/u_t)}, \tag{1.1}$$

where u_t and v_t denote the unemployment and vacancy rates, δ_t denotes the rate at which employed workers become unemployed (henceforth, the EU rate), and $A_t p(v_t/u_t)$ denotes the rate at which unemployed workers become employed (henceforth, the UE rate), which is given by the product between a parameter A_t that controls the efficiency of the search process and an increasing function $p(v_t/u_t)$ that controls the relationship between the tightness of the labor market.

The secular stability of the Beveridge curve is puzzling from the perspective of search theory. It suggests that either there has been no increase in the efficiency A_t of the search process

[1] Figures 1 and 2 are constructed using the data in Petrosky-Nadeau and Zhang (2014). We refer the reader to their paper for details about the construction of the vacancy rate series before World War II.

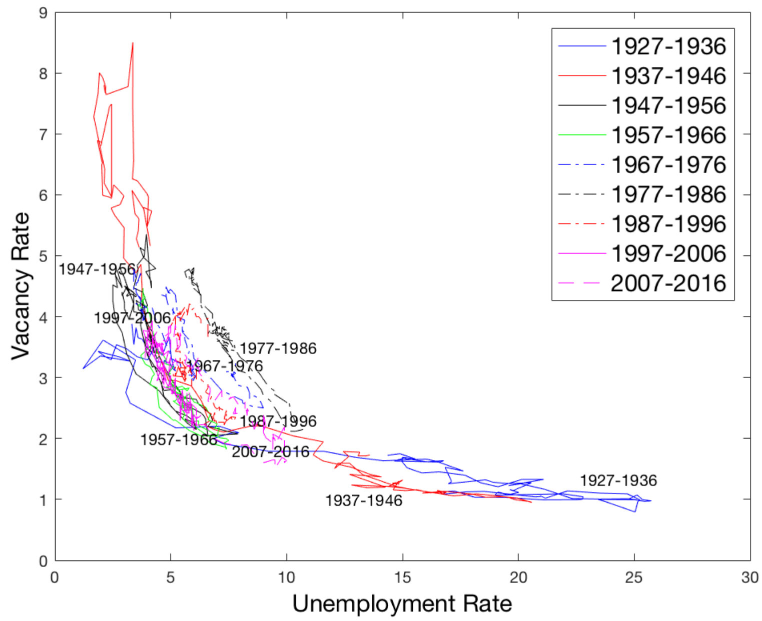

Figure 1: Beveridge Curve US: 1926-2018

from 1929 to 2018 or that every increase A_t has been offset by an increase in the EU rate δ_t. The first possibility seems unlikely, as the period from 1929 to 2018 has witnessed the development and diffusion of a great deal of information technology—the telephone, the fax machine, the copying machine, the computer, the Internet—which must have had an impact on the ability of firms to announce their job openings to the market, on the ability of workers to learn about and apply to job openings and, ultimately, on the efficiency A_t of the search process. The second possibility can be easily refuted by looking at the data. As illustrated in Figure 3, neither the UE rate nor the EU rate[2] display a clear upward secular trend.[3]

The aim of this paper is to explain why the unemployment rate, the UE rate, the EU rate, the vacancy rate and the Beveridge curve have all been substantially stable in the face of vast improvements in information technology. In a nutshell, our explanation is based on the observation that, while improvements in the information technology increase the rate at which workers learn about vacancies, they also make workers and firms more selective about the quality of the relationships that they are willing to establish. According to our explanation, the Beveridge curve is given by

$$u_t = \frac{\delta_t}{\delta_t + A_t p(v_t/u_t) F(R_t)}, \tag{1.2}$$

[2]The UE and EU rate are constructed as in Menzio and Shi (2011). The UE and EU rate corrected for time-aggregation as in Shimer (2005) are similar and also do not display any upward secular trend.

[3]A third possibility is that search frictions have nothing to do with unemployment, vacancies and the Beveridge curve. We do not pursue this line of inquiry.

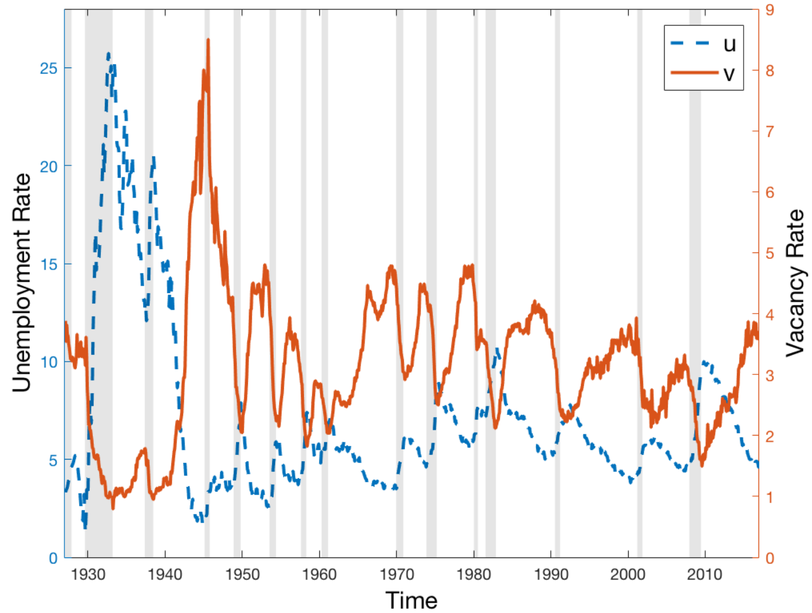

Figure 2: Unemployment and Vacancy Rates US: 1926-2018

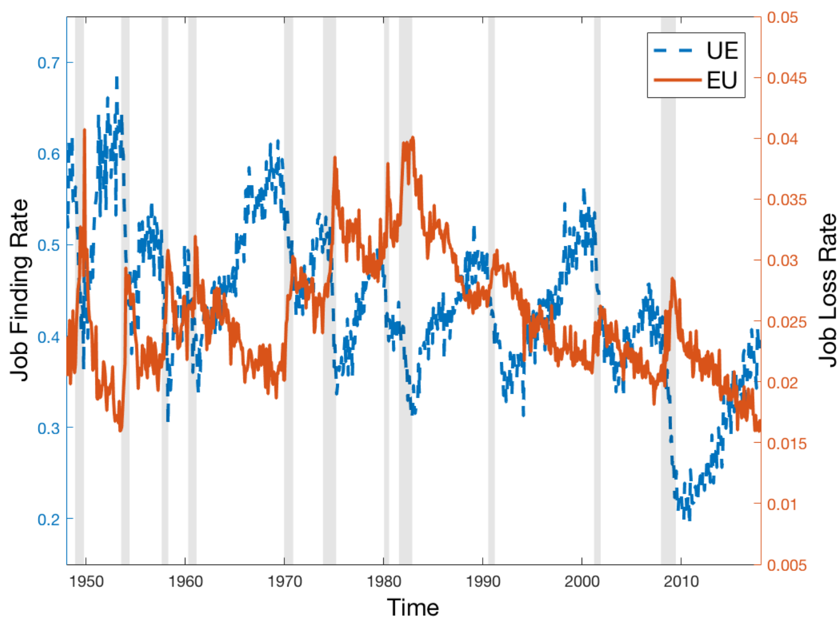

Figure 3: UE and EU Rates US: 1951-2008

where F is the distribution of quality of a firm-worker match and R_t is the reservation quality, i.e. an endogenous object that denotes the lowest quality such that a firm and a worker are willing to start a labor relationship. Under some conditions on the shape of F, the growth in the efficiency A_t of the search process is exactly offset by the endogenous decline in the probability $F(R_t)$ that the match is viable, thus leading to stationary unemployment and vacancy rates, UE and EU rates, and labor market tightness.

In the first part of the paper, we develop our theory in the context of a growth version of Mortensen and Pissarides (1994). As in Mortensen and Pissarides (1994), unemployed workers and vacant firms look for each other in the labor market and the rate at which workers contact firms is given by $A_t p(v_t/u_t)$. In contrast to Mortensen and Pissarides (1994), matches are "inspection goods," in the sense that—upon meeting—a firm and a worker observe the productivity of their match and, based on this information, decide whether to start an employment relationship or to keep on searching. The environment features growth in both the search technology and the production technology, as the efficiency of the search process grows at a constant rate g_A and the component of productivity that is common to all matches grows at the rate g_y.

We define a Balanced Growth Path (henceforth, BGP) as an equilibrium in which the unemployment rate, the UE rate, the EU rate, the vacancy rate and the tightness of the labor market are constant over time, while the cross-sectional distribution of workers across matches of different qualities grows at the endogenous, constant rate g_z, in the sense that every quantile of the distribution grows at the rate g_z. We find that a BGP exists if and only if the distribution F of the quality of a new matches is Pareto with some tail coefficient $\alpha > 1$, and the worker's income from unemployment and the firm's cost from posting a vacancy grow at the rate $g_y + g_A/\alpha$. In a BGP, unemployment, vacancies and the UE and EU rates are constant. The distribution of employed workers across matches of different qualities is a truncated Pareto that grows at the rate $g_z = g_A/\alpha$. The average productivity of labor, wages and output all grow at the rate $g_y + g_A/\alpha$.

The intuition behind our findings is simple. Improvements in the search technology lead to an increase in the rate at which unemployed workers meet vacant firms. Simultaneously, improvements in the search technology make firms and workers choosier about the quality of the matches that they are willing to form, as they make it easier to experiment with alternative partners. When the distribution of match qualities is Pareto, the two effects exactly cancel out, leading to a constant UE rate, EU rate and unemployment. The firm's return to filling a vacancy grows at the rate $g_A + g_y/\alpha$ because of improvements in the production and search technologies. If the cost of a vacancy grows at the same rate—because, say, opening a vacancy requires the use of labor—then the tightness of the labor market remains constant as well, and so does the vacancy rate. Interestingly, while improvements in the search technology do not lead to any decline in unemployment, they contribute to the growth rate of the economy with a strength that increases with the thickness $1/\alpha$ of the tail of the Pareto distribution of match qualities.

In the second part of the paper, we generalize the baseline model to allow workers to search the labor market not only when they are unemployed, but also when they have a job (albeit with potentially different intensity). The generalization of the model is relevant, as we know that workers move often from one job directly to another and, thus, the workers' opportunity cost of accepting a job out of unemployment is not to give up entirely on search. The analysis of the general model is harder, but the necessary and sufficient conditions for the existence of a BGP turn out to be the same as for the baseline model. Moreover, we show that, in any BGP of the on-the-job search model, unemployment, vacancies and the UE and EU rates remain constant over time, the distribution of employed workers across matches of different qualities is a truncated Fréchet that grows at the rate g_A/α, and the average productivity of labor, wages and output all grow at the rate $g_y + g_A/\alpha$.

In the third part of the paper, we further generalize the model to allow for growth in the size of the labor force and for non-constant returns to the scale of the labor market in the search process. We find that the conditions for the existence of a BGP are essentially the same as in the baseline model. We also find that, in the BGP, unemployment, vacancies and the UE and EU rates are constant. The distribution of employed workers across matches of different qualities grows at the rate $(g_A + \beta g_N)/\alpha$, where g_N is the growth rate of the labor force and β is the parameter than controls the returns to scale in the search process—with $\beta > 0$ meaning increasing, $\beta = 0$ constant, and $\beta < 0$ decreasing returns to scale. Finally, the average productivity of labor, wages and output per capita all grow at the rate $g_y + (g_A + \beta g_N)/\alpha$. The findings are intuitive, because non-constant returns to scale in the search process have the same type of effect on the rate at which workers meet firms as improvements in the search technology. The findings are interesting because they prove that—under the same condition which explain why unemployment and vacancies are constant in the face of an ever improving search technology—the returns to scale in the search process cannot be detected by looking at the relationship between unemployment rates (or UE rates) and the size of the labor market. Moreover, the ineffectiveness of this identification strategy applies both to the time-series and to the cross-section.

We conclude the paper with some observations on how to measure the contribution of declining search frictions to the growth rate of the economy and with some back-of-the-envelope calculations. In a BGP, the growth rate of the number of applicants considered for each vacancy—which is a measure of how selective firms and workers are—grows at the rate $g_A + \beta g_N$. In a BGP, the log of the ratio of the number of applicants considered for each vacancy in two labor markets is proportional to the log of the size of the two markets, and the constant of proportionality is β. Finally, in a BGP, the distribution of wages for workers hired out of unemployment is approximately a Pareto with a tail coefficient α. Thus, observations on applicants-per-vacancy over time and across space and observations on the cross-section of wages for homogeneous workers would be enough to identify the contribution g_A/α of improvements in the search technology to economic growth, the extent β of returns to scale in the search process, as well as the contribution $\beta g_N/\alpha$ of these returns to scale to economic growth.

As far as we know, there is no dataset that contains the secular evolution of applications-per-vacancy. However, the Employment Opportunity Pilot Project reports applications-per-vacancy for the US in 1981 (see Faberman and Menzio 2018) and several job sites, such as CareerBuilder.com and SnagAJob.com report applications-per-vacancy for the US in the 2010s (see Marinescu and Wolthoff 2016 and Faberman and Kudlyak 2016). Using these observations, we find that applications-per-vacancy grew between 1981 and 2011 by approximately 2% per year. Using applications-per-vacancy in different commuting zones of the US from CareerBuilder.com, we find that $\beta = 0.52$, which implies that applications-per-vacancy are 5.2% higher for vacancies located in a commuting zone that is 10% larger. The measurement of the growth rate of applications-per-vacancy implies that the contribution of declining search frictions to economic growth is 1.1% per year if $\alpha = 2$, 0.55% if $\alpha = 4$, and 0.275% if $\alpha = 8$. These are large numbers, even when the tail of the Pareto distribution of match qualities is very thin. The measurement of β—together with the fact that the US labor force has grown by 1.1% per year from 1981 to 2011–implies that 3/4 of the contribution of declining search frictions to economic growth is due to improvements in the search technology and 1/4 to increasing returns to scale. Moreover, the measurement of β implies that the productivity of labor in a commuting zone that is 10% larger is 2.5% higher if $\alpha = 2$, 1.25% higher if $\alpha = 4$ and 0.62% higher if $\alpha = 8$, just because of increasing returns in the search process.

The main contribution of the paper is to provide a theory to reconcile the search theory of unemployment with the observation that, in the face of vast improvements in information technology and, presumably, in the search technology, the unemployment rate, the UE rate, the EU rate, the vacancy rate and the Beveridge curve have all substantially remained stable in the US for nearly 100 years. The theory implies that, while improvements in the search technology do not affect unemployment, they do contribute to the growth of the economy. Moreover, some simple back-of-the-envelope calculations suggest that, indeed, the contribution of declining search frictions to economic growth is far from negligible.

From the methodological point of view, our paper belongs to the literature that seeks conditions on fundamentals under which an economy experiencing growth in the production technology features stationarity in some of its outcome (e.g., the labor share, the capital-output ratio, the interest rate, etc...). Key contributions in this literature include King, Plosser and Rebelo (1988) and, more recently, Grossman et al. (2007). Most of this literature focuses on Walrasian models. A few exceptions, which include Aghion and Howitt (1994) and Pissarides (2000), consider search-theoretic models of the labor market. Yet, these exceptions focus on understanding the interaction between the growth rate of the production technology and unemployment, rather than the effects of declining search frictions.

From the technical point of view, our paper is closely related to recent contributions in growth theory such as Perla and Tonetti (2014), Lucas and Moll (2014), Benhabib, Perla and Tonetti (2017) and Buera and Oberfield (2017). These papers focus on the diffusion of knowledge across individuals with different human capital and on the economic growth that might result from this process of diffusion. In these papers, as in ours, the key economic decision involves a choice between production and search. In our paper, search is for a new partner for the labor market. In these papers, search is for someone from whom to learn. Not surprisingly, in all of these papers as in ours, Pareto distributions play a key role for the existence of a BGP. Another paper that is technically similar to ours is Kortum (1997). This paper wants to rationalize why the growth rate of research output is constant in the face of an increasing fraction of labor devoted to research and development. This question is analogous to the one asked in our paper, namely why the unemployment rate is constant in the face of better and better information technology. Interestingly, the answer proposed by Kortum (1997) is conceptually similar to the one in our paper: while the rate of experimentation increases over time (where experimentation is research input in Kortum 1997 and search for a match in our paper), the probability of a successful experiments falls over time (where success is discovering a technology better than the status-quo in Kortum 1997 and finding a viable match in our paper).

From the substantive point of view, our paper is related to the fundamental idea that lower trading frictions allow agents to become more and more specialized. Kiyotaki and Wright (1993) make this point in the context of a search theoretic model of the product market. They show that the introduction of fiat money effectively reduces trading frictions and allows agents to produce more specialized goods. Ellison and Ellison (2018) show that the reduction of trading frictions brought about by the Internet has led to better matching between products and consumers and, in doing so, to an increase in consumer surplus. These papers make the same fundamental point as ours, although they only examine a one-time rather than a continuous decline in search frictions. Moreover, these papers focus on search frictions in the product rather than in the labor market. To the best of our knowledge, ours is the first paper that analyzes the effect of declining search frictions in the labor market and tries to quantify its effect on growth.

An immediate corollary of our theory about the independence of unemployment, vacancies and UE and EU rates from the efficiency of the search technology in the time-series is that these variables will also be independent from the size of the labor market in the cross-section, even in the presence of increasing returns to scale in the search process. Thus, the same theory that explains a time-series phenomenon also explains why, in the data, unemployment and the job-finding rate are uncorrelated with the size of the labor market.[4] Moreover, the same strategy that can be used to identify the contribution of declining search frictions to productivity in the time-series (i.e., measuring applicants-per-vacancy) can be used to identify the extent of returns to scale in the search process and its contribution to the city-size wage premium. Our preliminary implementation of this identification strategy suggests that, indeed, the search process features strong increasing returns to scale and that these returns to scale contribute to a non-negligible fraction of the city-size wage premium. The idea that an increase in how selective firms and workers are may hide the extent of increasing returns to search is not entirely novel. Petrongolo and Pissarides (2006) show—in the context of a partial equilibrium model—that changes in the workers' reservation wage partially offset the effect of market size on the job-finding rate. Using survey data on self-reported reservation wages, they show that, in fact, reservation wages are systematically higher in larger markets. Gautier and Teulings (2009) argue that increasing returns to scale in search may be further offset by the endogenous composition of workers in larger and smaller cities. Our theory is similar in spirit, but it ties together time-series and cross-sectional facts. Moreover, our analysis is focused on finding the conditions under which the increase in selectivity exactly neutralizes the effect of the decline in search frictions on the job-finding rate, rather than simply dampening such effect.

2 Basic model

In this section, we study a very simple search-theoretic model of the labor market in the spirit of Mortensen and Pissarides (1994) where both the production technology—as captured by the component of productivity that is common to all firm-worker matches—and the search technology—as captured by the efficiency of the matching function—improve over time at a constant rate and firm-worker matches are inspection goods—in the sense that the component of productivity that is idiosyncratic to a match is observed as soon as a firm and a worker meet. We look for conditions under which there exists an equilibrium such that, even though search frictions get smaller and smaller over time, the unemployment rate, the vacancy rate, and the rate at which workers move in and out of unemployment all remain constant. We refer to this as a Balanced Growth Path (henceforth, BGP) equilibrium. We show that a BGP exists if and only if the distribution of idiosyncratic match productivities is Pareto and the cost of a vacancy as well as the flow income of unemployment grow at the same rate as the economy. The growth rate of the economy depends on the growth rate of the production technology, the growth rate

[4] Clearly, another possibility is that unemployment is independent from the size of the labor market because the returns to scale in the search process are constant. This possibility is at odds with our observation that applicants-per-vacancy increase with the size of the market and with the observation of Petrongolo and Pissarides (2006) that the workers' reservation wage increases with the size of the market.

of the search technology and on the tail coefficient of the Pareto distribution of match qualities.

2.1 Environment

The labor market is populated by a continuum of ex-ante identical workers with measure 1. Each worker's objective is to maximize the present value of his labor income discounted at the rate $r > 0$, where labor income is given by a wage w_t if the worker is employed and by an unemployment benefit b_t if the worker is unemployed. The labor market is also populated by a continuum of ex-ante identical firms with some positive measure. Each firm is infinitely lived and operates a technology that turns the flow of labor input from a worker into a flow of $y_t z$ units of output, where y_t is the component of labor productivity that is common to all firm-worker pairs at date t and z is a component of productivity that is specific to a particular firm-worker pair. Each firm's objective is to maximize the present value of its profit discounted at the rate r.

The labor market is subject to search frictions. In particular, workers need to spend time searching the market to locate firms with vacant jobs. Firms need to spend resources to advertise job vacancies and locate workers. We assume that workers can only search if unemployed. We assume that firms have to pay a flow cost of k_t units of output to keep a vacancy open. We denote as u_t the measure of unemployed workers at date t and with v_t the measure of vacant jobs at date t. The outcome of the search process is a flow $A_t M(u_t, v_t)$ of bilateral meetings between unemployed workers and vacant jobs, where A_t is a measure of the efficiency of the search process at date t and $M(u, v)$ is a constant returns to scale function.[5] The outcome of the search process implies that an unemployed worker meets a vacancy at the Poisson rate $A_t p(\theta_t)$, where θ_t denotes the tightness v_t/u_t of the labor market and $p(\theta) = M(1, \theta)$ is such that $p(0) = 0$, $p'(\cdot) > 0$, $p''(\cdot) < 0$ and $p(\infty) \to \infty$. Similarly, the outcome of the search process implies that a vacant job meets a worker at the Poisson rate $A_t q(\theta_t)$, where $q(\theta) = p(\theta)/\theta$ is such that $q(0) \to \infty$, $q'(\cdot) < 0$ and $q(\infty) = 0$.

Upon meeting, a firm and a worker observe the idiosyncratic component z of their productivity. The idiosyncratic component z is drawn from a cumulative distribution function F, and we denote as $\overline{F}$ the survival probability $1 - F$. Based on the observation of z, the firm and the worker decide whether to match or not. If they do, the worker becomes employed and starts producing a flow of $y_t z$ units of output. If they do not, the worker remains unemployed and keeps searching for some other job, and the firm returns to the labor market and keeps advertising its vacancy to find some other worker. Once matched, a firm and a worker continue producing until

[5]We assume that search is random. The assumption is not important for establishing the conditions for the existence of a BGP. Indeed, it is straightforward to show that the same conditions apply to a model with directed search like Moen (1997) if workers search only when unemployed or like Menzio and Shi (2010, 2011) if workers search off and on the job. We decided to use random search because of its popularity. However, directed search would allow us to solve for the equilibrium dynamics outside of the BGP.

their relationship is dissolved.[6]

The terms of the employment contract between a firm and a worker are determined according to the axiomatic Nash bargaining solution, where the worker's threat point is the value of being unemployed and the firm's threat point is the value of a vacant job. The worker's bargaining power is γ and the firm's bargaining power is $1-\gamma$, with $\gamma \in (0,1)$. We assume that the contingencies of the employment contract are rich enough so that the sum of the value to the worker and to the firm from being matched is maximized. The joint value of the firm-worker match is clearly maximized if the employment contract can specify both a wage path and a time when the relationship shall be dissolved. The joint value of the match is also maximized if the employment contract can specify only a wage path, while the decision of dissolving the relationship is left out of the contract and can be made unilaterally by either party. Indeed, the bargained wage path will assign a positive fraction of the gains from trade to both the worker and the firm and, hence, the two parties will separate only when the gains from trade are exhausted. Finally, the joint value of the match is maximized even if, as in Mortensen and Pissarides (1994), an employment contract can only specify the wage over the next dt units of time and it is then renegotiated.

We assume that both production and search technologies improve over time. The aggregate component of labor productivity grows at the rate g_y, i.e. $y_t = y_0 e^{g_y t}$. The assumption is meant to capture the idea that progress in the production technologies allows to generate more output using the same amount of labor. The efficiency of the meeting function grows at the rate $g_A > 0$, i.e. $A_t = A_0 e^{g_A t}$. The assumption is meant to capture the idea that progress in the communication and information technology facilitates search and leads to more meetings between the same number of unemployed workers and vacant firms. We also assume that the flow cost of a vacancy grows at some constant rate g_k and that the flow unemployment benefit grows at the rate g_b, i.e. $k_t = k_0 e^{g_k t}$ and $b_t = b_0 e^{g_b t}$.

Our model is a version of Mortensen and Pissarides (1994) in which there is growth in the production and search technologies and in which firm-worker matches are inspection goods—in the sense that the component of productivity that is idiosyncratic to a match is observed before the firm and the worker decided to form the match or not. In a model like ours where the search technology improves over time, the rate at which unemployed workers meet vacancies is ever growing. For the rate at which unemployed workers become employed to remain constant, there needs to be some countervailing selection mechanism that can lead to a constant decline in the probability that a meeting between a worker and a vacancy turns into a match. A natural way to create endogenous selectivity is to assume that matches are heterogeneous. The assumption captures the idea that workers and vacancies are heterogeneous and that such heterogeneity

[6]We assume that an employment relationship is only broken up by choice. It would be straightforward to generalize the model to allow the employment relationship to also break up for exogenous reasons.

affects the quality of a particular worker in a particular vacancy. If the quality of a firm-worker pair was observed after the match was formed (i.e., if matches were experience goods), then the mechanism would be moot, as any time a worker would meet a vacancy he would become employed, at least temporarily. Therefore, it is necessary to assume that the quality of a firm-worker pair can be observed before the match is formed. In our model, we assume that the quality is perfectly observed. However, the mechanism would still be operative if the firm and the worker only observed a signal of quality (as in, e.g., Menzio and Shi 2011).

2.2 Definition of BGP

In order to formally define a BGP, we need to introduce some additional notation. We let U_t denote the present value of labor income for a worker who is unemployed at date t. We let $E_t(z|w,d)$ and $J_t(z|w,d)$ denote the present value of labor income and the present value of profits for a worker and a firm who, at date t, form a match of quality z and agree to an employment contract in which the wage is w_{t+x} at date $t+x$ and the break-up date is $t+d$. We denote as $V_t(z|w,d)$ the joint value of the firm-worker match, which is defined as the sum of $E_t(z|w,d)$ and $J_t(z|w,d)$. We denote as $S_t(z)$ the surplus of the firm-worker match, defined as the difference between the maximized joint value of the match and the worker's outside option. Finally, we denote as u_t the measure of workers who are unemployed at date t and $G_t(z)$ the measure of workers who are employed in a match of quality non-greater than z at date t.

Given an employment contract (w,d), the present value of income and the present value of profits for a worker and a firm in a match of quality z at date t are, respectively, given by

$$E_t(z|w,d) = \int_0^d e^{-rx} w_{t+x} dx + e^{-rd} U_{t+d}, \tag{2.1}$$

$$J_t(z|w,d) = \int_0^d e^{-rd} (y_{t+x} z - w_{t+x}) dx. \tag{2.2}$$

The above expressions are easy to understand. At date $t+x$, the worker's labor income is w_{t+x}. At date $t+d$, the match breaks up and the worker's continuation present value of income is U_{t+d}. Similarly, at date $t+x$, the firm's profit is $y_{t+x}z - w_{t+x}$. At date $t+d$, the match breaks up and the firm's continuation present value of profits is zero. The joint value of the match between the firm and the worker is given by

$$V_t(z|d) = \int_0^d e^{-rx} y_{t+x} z dx + e^{-rd} U_{t+d}. \tag{2.3}$$

Note that, since the wage transfers utility at the rate of 1 to 1 from the firm to the worker, the joint value of the match depends only on the break up date d and not on the wage path. Clearly, $V_t(z|d)$ is well-defined for an arbitrary d only if the discount factor exceeds the growth rate of the common component of productivity, i.e. $r > g_y$.

The employment contract (w,d) signed by the firm and the worker upon meeting is such that the break-up date d maximizes the joint value of the match. The break-up date d^* maximizes the joint value of the match only if

$$y_{t+d^*}z + \mathring{U}_{t+d^*} \leq rU_{t+d^*}, \text{ and } d^* \geq 0, \tag{2.4}$$

where the two inequalities hold with complementary slackness. The necessary condition for optimality (2.4) has a simple interpretation. The marginal benefit of delaying the break-up of the match is given by the flow of output $y_{t+d^*}z$ plus the time derivative of the worker's present value of income in unemployment. The marginal cost of delaying the break-up of the match is given by the annuitized sum of the present values that the worker and the firm can attain by being single, which is rU_{t+d^*}. Condition (2.4) then states that either $d^* = 0$ and the marginal cost of delaying the break-up exceeds the marginal benefit, or $d^* > 0$ and the marginal cost of delaying the break-up equals the marginal benefit. The necessary condition in (2.4) is also sufficient if the marginal cost of delaying the break-up (the term on the right-hand side) increases more rapidly than the marginal benefit (the term on the left-hand side). This must be true in any BGP, as otherwise the reservation quality R_t which we define below would decline and, hence, the UE rate would increase over time.

We define the reservation quality R_t as

$$R_t = \frac{1}{y_t}\left(rU_t - \mathring{U}_t\right). \tag{2.5}$$

Since the optimality condition (2.4) is both necessary and sufficient, it follows from the definition of R_t that: (i) for all $z \leq R_t$, the optimal break-up time d^* is equal to 0 and the joint value of the match is $V_t(z|d^*) = U_t$; (ii) for all $z > R_t$, the optimal break-up time d^* is strictly positive and $V_t(z|d^*) > U_t$. These observations imply that, upon meeting, a firm and a worker form a match and start producing if and only if they observe a match quality $z > R_t$. Otherwise, the firm and the worker keep searching.

Conditional on observing a match quality $z > R_t$, the employment (w,d) signed by the firm and the worker is such that the Nash product $(E_t(z|w,d^*) - U_t)^\gamma J_t(z|w,d^*)^{1-\gamma}$ is maximized. Since the wage transfers utility at the rate of 1 to 1 from the firm to the worker, the wage path that maximizes the Nash product is such that the worker captures a fraction γ of the surplus of the match, while the firm captures a fraction $1-\gamma$ of it. That is,

$$\begin{aligned} E_t(z|w^*,d^*) &= U_t + \gamma S_t(z), \\ J_t(z|w^*,d^*) &= (1-\gamma)S_t(z), \end{aligned} \tag{2.6}$$

where the surplus is defined as

$$S_t(z) = V_t(z|d^*) - U_t. \tag{2.7}$$

The present value of income for an unemployed worker satisfies

$$rU_t = b_t + A_t p(\theta)\gamma \int_{R_t} S_t(\hat{z}) dF(\hat{z}) + \mathring{U}_t. \tag{2.8}$$

The left-hand side of (2.8) is the annuitized present value of income for an unemployed worker. The right-hand side is the sum of three terms. The first term is the flow of income to a worker when he is unemployed. The second term is the annuitized value of searching to an unemployed worker, which is given by the rate at which a worker meets a firm times the expected increase in the lifetime income of a worker upon meeting a firm. In light of (2.6), the expected increase in the lifetime income of a worker upon meeting a firm is equal to a fraction γ of the surplus if $z > R_t$ and zero otherwise. The last term in (2.8) is the time derivative of the worker's present value of income in the state of unemployment. Clearly, U_t is well-defined only if the discount rate is greater than the growth rate of the unemployment income, i.e. $r > g_b$.

From the definition of $S_t(z)$ in (2.7) and the expressions for $V_t(z|d^*)$ and U_t in (2.3) and (2.8), it follows that the surplus of a firm-worker match of quality $z > R_t$ satisfies

$$rS_t(z) = y_t z - b_t - A_t p(\theta)\gamma \int_{R_t} S_t(\hat{z}) dF(\hat{z}) + \mathring{S}_t(z). \tag{2.9}$$

The left-hand side of (2.9) is the annuitized surplus of a firm-worker match. The right-hand side is the sum of three terms. The first term is the difference between the flow income produced by the firm-worker match and the flow of income of an unemployed worker. The second term is the annuitized value of searching to an unemployed worker. The last term is the time derivative of the surplus of the match.

The tightness of the labor market satisfies

$$k_t = A_t q(\theta)(1-\gamma) \int_{R_t} S_t(\hat{z}) dF(\hat{z}) \tag{2.10}$$

The left-hand side of (2.10) is the cost to the firm of maintaining a vacancy at date t. The right-hand side is the rate at which the firm meets a worker times the expected present value of profits to the firm conditional on meeting a worker. In light of (2.6), the expected present value of profits to the firm upon meeting a worker is equal to a fraction $1-\gamma$ of the surplus if $z > R_t$ and zero otherwise. Equilibrium condition (2.10) then states that the tightness of the labor market equates the cost and the benefit to the firm from keeping a vacancy open at date t and that such tightness is constant over time.

The rate at which unemployed workers become employed (UE rate), the rate at which employed workers become unemployed (EU rate) and the measure of unemployed workers are stationary if and only if

$$\begin{aligned} A_t p(\theta)(1-F(R_t)) &= h_{UE}, && (2.11)\\ G_t'(R_t)\mathring{R}_t &= h_{EU}, && (2.12)\\ uh_{UE} &= (1-u)h_{EU}. && (2.13) \end{aligned}$$

The expression on the left-hand side of (2.11) is the UE rate at date t, which is given by the rate at which an unemployed worker meets a firm at time t times the probability that the quality z of the firm-worker match is above the reservation threshold R_t. Condition (2.11) requires the UE rate to be constant at some level h_{UE}. The expression on the left-hand side of (2.12) is the EU rate at date t, which is given by product between the density of the date-t distribution $G_t(z)$ of employed workers across different match qualities evaluated at the reservation threshold R_t and the time-derivative of R_t. Condition (2.12) requires the EU rate to be constant at some level h_{EU}. Finally, condition (2.13) states that the measure u of unemployed workers is constant over time if and only if the flow of workers out of unemployment (the left-hand side of (2.13)) equals the flow of workers into unemployment (the right-hand side of (2.13)).

In contrast to the measure u of unemployed workers, the distribution $G_t(z)$ of employed workers across matches of different quality z cannot be constant over time in a BGP. Indeed, in any BGP, R_t must increases over time to keep the UE rate constant and R_t is the lower bound on the support of the distribution G_t. Instead, in a BGP, G_t grows at a constant rate, in the sense that every quantile of the distribution grows at the same, constant rate g_z. Formally, in a BGP $z_t(x) = z_0(x)e^{g_z t}$ for all $x \in [0,1]$ and $t \geq 0$, where $z_t(x)$ denotes the xth quantile of G_t.

The condition $z_t(x) = z_0(x)e^{g_z t}$ is satisfied if and only if

$$\begin{aligned} & (1-u)\left[G_t(z_t(x)e^{g_z dt}) - G_t(z_t(x))\right] + uA_t p(\theta)\left[F(z_t(x)e^{g_z dt}) - F(R_t e^{g_z dt})\right] dt \\ = \; & (1-u)\left[G_t(R_t e^{g_z dt}) - G_t(R_t)\right]. \end{aligned} \tag{2.14}$$

The left-hand side of (2.14) is the flow of workers into matches of a quality z that is below the xth quantile, which is given by the sum of two terms. The first term is the measure of workers who, at date t, are employed in a match of quality z just above the xth quantile and who fall below the xth quantile in the next dt units of time. The second term is the measure of workers who, at date t, are unemployed and find a job of quality z below the xth quantile in the next dt units of time. The right-hand side of (2.14) is the flow of workers out of matches of a quality z that is below the xth quantile, which is given by the measure of workers who, at date t, are employed in a match of quality z just above the reservation level R_t and who, over the next dt units of time, move into unemployment. Dividing both sides of (2.14) by dt and taking the limit for $dt \rightarrow 0$, we obtain

$$(1-u)G_t'(z_t(x))z_t(x)g_z + uA_t p(\theta)\left[F(z_t(x)) - F(R_t)\right] = (1-u)G_t'(R_t)R_t g_z. \tag{2.15}$$

We are now in the position to formally define a BGP.

Definition 1: *A BGP is a tuple $\{R_t, U_t, S_t, \theta, h_{UE}, h_{EU}, u, G_t\}$ such that for all $t \geq 0$: (i) R_t, U_t and S_t satisfy (2.5), (2.8) and (2.9); (ii) θ satisfies (2.10); (iii) h_{UE}, h_{EU} and u satisfy (2.11), (2.12) and (2.13); (iv) G_t satisfies (2.15).*

2.3 Necessary conditions for a BGP

In this subsection, we derive some restrictions on the fundamentals of the model that are necessary for the existence of a BGP. First, we derive a necessary condition on the distribution of match qualities F. To this aim, note that the stationarity condition (2.11) for the UE rate implies that the time-derivative of $A_t p(\theta)(1-F(R_t))$ must be equal to zero, i.e.

$$\mathring{A}(t)[1-F(R_t)] - A_t F'(R_t)\mathring{R}_t = 0, \ \forall t \geq 0. \tag{2.16}$$

The efficiency A_t of the matching function grows at the constant rate g_A. The reservation quality R_t grows at the constant rate g_z, because it is the quantile zero of the distribution G_t of employed workers across matches of different qualities. In light of these observations, we can write (2.16) as

$$\frac{F'(R_t)R_t}{1-F(R_t)} = \frac{g_A}{g_z}, \ \forall R_t \geq R_0. \tag{2.17}$$

The expression in (2.17) is a differential equation for F. The solution to this differential equation[7] that satisfies the boundary condition $F(\infty) = 1$ is

$$F(z) = 1 - \left(\frac{z_\ell}{z}\right)^\alpha, \tag{2.18}$$

where $\alpha = g_A/g_z$ and z_ℓ is an arbitrary lower bound non-greater than R_0. Therefore, a BGP may only exist if the distribution F of match qualities has the shape given by (2.18), which is a Pareto with coefficient α. We will assume that the lower bound z_ℓ of the distribution F is smaller than b_0/y_0, which guarantees that $z_\ell < R_0$ since $R_0 > b_0/y_0$. Moreover, in any BGP, the growth rate g_z of the distribution of employed workers across matches of different qualities must be equal to the ratio between the growth rate g_A of the matching function efficiency and the coefficient α of the Pareto distribution F.

Next, we derive necessary conditions on the growth rate g_k of the firm's vacancy cost and on

[7] Formally, (2.17) is a differential equation for $F(z)$ for all $z \geq R_0$. Since the shape of the distribution F below R_0 is inconsequential, as it does not affect any decisions, we can assume without loss in generality that (2.17) holds for all z.

the growth rate g_b of the worker's unemployment income. To this aim, note that the reservation quality R_t must satisfy the equilibrium condition (2.5). Using the equilibrium condition (2.10) for the tightness θ of the labor market and the equilibrium condition (2.8) for the value of unemployment U_t, we can rewrite (2.5) as

$$R_t = \frac{b_t}{y_t} + \frac{\gamma}{1-\gamma}\frac{\theta k_t}{y_t}, \forall t \geq 0. \tag{2.19}$$

In the above expression, the reservation quality R_t grows at the rate g_z, the worker's income from unemployment b_t grows at the rate g_b, the firm's vacancy cost k_t grows at the rate g_k, and the common component of productivity y_t grows at the rate g_y. From these observations, it follows that the time-derivative of (2.19) is

$$g_z = (g_b - g_y) + (g_k - g_b) \cdot \frac{\gamma k_0 \theta}{(1-\gamma) b_0 \exp((g_b - g_k)t) + \gamma k_0 \theta}, \forall t \geq 0. \tag{2.20}$$

Condition (2.20) holds only if $g_k = g_b$ and $g_b = g_y + g_z = g_y + g_A/\alpha$. Therefore, a BGP may only exist if the growth rate g_b of the worker's unemployment income and the growth rate g_k of the firm's vacancy cost are equal to the sum between the growth rate g_y of the common component of productivity and the growth rate g_A/α of the efficiency of the matching function divided by the coefficient α of the Pareto distribution F.

We summarize our findings in the following proposition.

Proposition 1 *(Necessary conditions for a BGP). Consider arbitrary growth rates $g_y > 0$ and $g_A > 0$ for the production and the search technologies.*

(i) A BGP may exist only if: (a) the match quality distribution F is a Pareto with coefficient α; (b) the growth rates of the worker's unemployment income and of the firm's vacancy cost, g_b and g_k, are both equal to $g_y + g_A/\alpha$; (c) the discount rate r is greater than $g_y + g_A/\alpha$.

(ii) In any BGP, the growth rate g_z for the distribution of employed workers across matches equals g_A/α.

A couple of comments about Proposition 1 are in order. First, let us give some intuition for why a BGP can only exist if the distribution F of match qualities is Pareto. The rate at which an unemployed worker meets a firm grows at the rate g_A. Therefore, for the rate at which an unemployed worker to become employed to be constant, the probability that a meeting turns into a match must decline at the rate g_A. The probability that a meeting turns into a match is the probability that the match quality exceeds the reservation R_t, which grows at the rate g_z. The only distribution function with the property that the measure of realizations above a cutoff that grows at the rate g_z falls at the rate g_A is a Pareto distribution with coefficient $\alpha = g_A/g_z$.

Second, let us discuss the conditions on g_b and g_k that are necessary for the existence of a BGP. The tightness of the labor market equates the cost k_t of opening a vacancy to the benefit, which is proportional to the rate $A_t q(\theta)$ at which a firm meets a worker and to the expected surplus of a meeting. In turn, the surplus of a meeting is proportional to the difference between the reservation quality $R_t y_t$ and the unemployment income b_t. These observations imply that the tightness of the labor market can be constant over time only if the cost of a vacancy grows at the same rate as the unemployment income. The common growth rate must be the growth rate of $R_t y_t$, which is $g_y + g_z$. At first blush, there seems to be no reason why the cost of a vacancy and the unemployment benefit would grow at precisely the rate $g_y + g_z$. However, as we shall see in Section 2.4, $g_y + g_z$ is also the growth rate of average labor productivity, wages and aggregate output. Hence, if the benefit b_t is proportional to wages, it would grow at the rate $g_y + g_z$. Similarly, if opening a vacancy requires labor, k_t would also grow at the rate $g_y + g_z$.

Finally, let us discuss the condition $r > g_y + g_A/\alpha$. As noted in the previous subsection, $r > g_y$ is necessary to guarantee that V_t is well-defined, and $r > g_b$ is necessary to guarantee that U_t is well-defined. Since $g_b = g_y + g_A/\alpha$ and $g_b > g_y$, these necessary conditions are equivalent to $r > g_y + g_A/\alpha$. This condition is very intuitive, as it requires that the discount rate is greater than the growth rate of the economy.

2.4 Existence and uniqueness of a BGP

Let us assume that the distribution F of match qualities is Pareto with coefficient α, the growth rate g_b of the unemployment income b_t is $g_y + g_A/\alpha$, and the growth rate g_k of the vacancy cost k_t is also $g_y + g_A/\alpha$. Let us also assume that the discount rate r is greater than $g_y + g_A/\alpha$. In light of Proposition 1, a BGP cannot exist if these assumptions about the fundamentals do not hold. Given these assumptions, we use the equilibrium conditions to construct a candidate BGP.

The surplus $S_t(z)$ for a match of quality $z > R_t$ is given by equilibrium condition (2.9), which can be written as

$$S_t(z) = \int_0^{d^*} e^{-rx} y_{t+x}(z - R_{t+x})dx. \tag{2.21}$$

The above expression makes use of the equilibrium condition (2.5) for the reservation quality R_t and of the fact that the surplus of the match is 0 at date $t + d^*$, with d^* given by $z = R_{t+d^*}$. Intuitively, the surplus of a match of quality z is the present discounted value of the difference between the flow of output generated by the match of quality z and the flow of output generated in a match of quality R_{t+x}. Using the fact that $y_{t+x} = y_t e^{g_y x}$ and $R_{t+x} = R_t e^{g_z x}$ we can solve the integral in (2.21) and obtain

$$S_t(z) = y_t \left\{ \frac{z}{r - g_y} \left[1 - \left(\frac{R_t}{z} \right)^{\frac{r-g_y}{g_z}} \right] - \frac{R_t}{r - g_y - g_z} \left[1 - \left(\frac{R_t}{z} \right)^{\frac{r-g_y-g_z}{g_z}} \right] \right\}. \tag{2.22}$$

The expected surplus of a match between a firm and a worker at date t, which we shall denote as S_t^e, is given by

$$S_t^e = \alpha \int_{R_t} S_t(z) \left(\frac{z_\ell}{z}\right)^\alpha \frac{1}{z} dz, \tag{2.23}$$

where the above expression makes use of the fact that the distribution F is Pareto with coefficient α and the surplus $S_t(z)$ of a match of quality z is zero for all $z < R_t$. After replacing $S_t(z)$ with the right-hand side of (2.22) and solving the integral, we find that, as long as $\alpha > 1$, the expected surplus of a match exists and is equal to

$$S_t^e = \Phi y_t R_t^{-(\alpha-1)}, \tag{2.24}$$

where Φ is a strictly positive constant that depends only on parameters.[8] If the condition $\alpha > 1$ is violated, the expected surplus of a match in (2.24) is not well-defined. The condition $\alpha > 1$ is intuitive, as it is equivalent to the condition for the Pareto distribution F to have a finite expected value.

The reservation quality R_t is given by the equilibrium condition (2.5). Using the equilibrium condition (2.8) for U_t and of the solution for S_t^e in (2.24), we can rewrite (2.5) as

$$R_t = \frac{b_t}{y_t} + A_t p(\theta) \gamma \Phi R_t^{-(\alpha-1)}. \tag{2.25}$$

When evaluated at date $t = 0$, condition (2.25) holds if and only if the reservation quality R_0 is

$$R_0 = \frac{b_0}{y_0} + A_0 p(\theta) \gamma \Phi R_0^{-(\alpha-1)}. \tag{2.26}$$

When evaluated at date $t > 0$, condition (2.25) holds if and only if it holds at $t = 0$ and the left-hand side grows at the same rate as the right-hand side. The growth rate of the left-hand side is g_z. The growth rate of the first term on the right-hand side is $g_b - g_y$, which is equal to g_A/α. The growth rate of the second term is $g_A - (\alpha - 1)g_z$. These growth rates are equal if and only if g_z is equal to g_A/α.

The tightness θ of the labor market is given by the equilibrium condition (2.10). Using the solution for S_t^e in (2.26), we can rewrite (2.10) as

$$k_t = A_t q(\theta)(1-\gamma)\Phi y_t R_t^{-(\alpha-1)}. \tag{2.27}$$

[8]Formally, Φ is defined as

$$\Phi = \alpha z_\ell^\alpha \left\{ \left[\frac{1}{r - g_y - g_z} - \frac{1}{r - g_y}\right] \frac{g_z}{r + (\alpha-1)g_z - g_y} + \frac{1}{(\alpha-1)(r-g_y)} - \frac{1}{\alpha(r - g_y - g_z)} \right\}.$$

It is a matter of simple algebra to show that Φ is strictly positive as long as $r > g_y + g_z$, which is the case as we assumed that $r > g_y + g_A/\alpha$ and, in any BGP, $g_z = g_A/\alpha$.

When evaluated at date $t = 0$, (2.27) holds if and only if the market tightness θ is such that

$$k_0 = A_0 q(\theta)(1-\gamma)\Phi y_0 R_0^{-(\alpha-1)}. \tag{2.28}$$

When evaluated at $t > 0$, (2.27) holds if and only if it holds at date $t = 0$ and the growth rate g_k of the vacancy cost k_t, which is the growth rate of the left-hand side, is equal to $g_A + g_y - (\alpha-1)g_z$, which is the growth rate of the right-hand side. This is the case as $g_k = g_y + g_A/\alpha$ and $g_z = g_A/\alpha$.

The distribution $G_t(ze^{g_z t}) = G_0(z)$ of employed workers across matches of different qualities must satisfy the inflow-outflow condition (2.15). Since stationarity condition (2.13) for unemployment implies that the flow of workers into unemployment $(1-u)G_t'(R_t)R_t g_z$ is equal to the flow out $uA_t p(\theta)[1-F(R_t)]$, we can rewrite (2.15) as

$$(1-u)\left[G_t'(ze^{g_z t})ze^{g_z t}g_z\right] = uA_t p(\theta)\left(\frac{z_\ell}{ze^{g_z t}}\right)^{\alpha}. \tag{2.29}$$

We first use the expression in (2.29) to recover the initial distribution G_0 of employed workers across matches of different qualities. To this aim, note that (2.29) at $t = 0$ is a differential equation for G_0. The solution to the differential equation that satisfies the boundary condition $G_0(\infty) = 1$ is

$$G_0(z) = 1 - \frac{uA_0 p(\theta)}{(1-u)g_A}\left(\frac{z_\ell}{z}\right)^{\alpha}. \tag{2.30}$$

As $G_0(R_0)$ must be equal to zero, (2.30) implies that the unemployment rate u must be

$$u = \frac{g_A}{g_A + A_0 p(\theta)(z_\ell/R_0)^{\alpha}}. \tag{2.31}$$

Using (2.31), we can rewrite (2.30) as

$$G_0(z) = 1 - \left(\frac{R_0}{z}\right)^{\alpha}. \tag{2.32}$$

That is, the initial distribution G_0 of employed workers across matches of different qualities is the sampling distribution F truncated at the reservation quality R_0.

We then need to verify that the balanced growth distribution $G_t(ze^{g_z t}) = G_0(z)$ satisfies (2.29) for all $t \geq 0$. This is the case if and only if G_0 is given by (2.30) and the left-hand side of (2.29) grows at the same rate as the right-hand side of (2.29). The left-hand side of (2.29) grows at the rate of zero, as $z_t(x)$ grows at the rate g_z and $G_t'(z_t(x))$ grows at the rate $-g_z$. The right-hand side grows at the rate 0 as well, since $g_A - g_z\alpha = 0$.

Finally, we need to solve for the UE and EU rates. The conditions (2.11) and (2.12) for the

UE and EU rates at date $t=0$ can be written as

$$h_{UE} = A_0 p(\theta)\left(\frac{z_\ell}{R_0}\right)^{\alpha}, \quad (2.33)$$

$$h_{EU} = G_0'(R_0)R_0 g_z = \alpha g_z = g_A. \quad (2.34)$$

The above value for h_{UE} satisfies condition (2.11) for all $t \geq 0$, as A_t grows at the rate g_A and $(z_\ell/R_t)^{\alpha}$ grows at the rate $-\alpha g_z = -g_A$. The above value for h_{EU} satisfies condition (2.12) for all $t \geq 0$, as $G_t'(R_t)$ grows at the rate $-g_z$ and R_t grows at the rate g_z.

We are now in the position to establish the uniqueness and existence of a BGP. The values of R_0 and θ must solve the system of two equations given by (2.26) and (2.28). The solution of (2.26) with respect to R_0 exists and is unique for all values of $\theta \geq 0$, and we denote it as $\psi_1(\theta)$. It is straightforward to verify that $\psi_1(0) = b_0/y_0$, $\psi_1'(\theta) > 0$ and $\psi_1(\infty) \to \infty$. The solution of (2.28) with respect to R_0 also exists and is unique for all values of $\theta \geq 0$, and we shall denote it as $\psi_2(\theta)$. It is straightforward to verify that $\psi_2(0) \to \infty$, $\psi_2'(\theta) < 0$, $\psi_2(\infty) = 0$. From these observations, it follows immediately that there is a unique pair (R_0, θ) with $R_0 > 0$ and $\theta > 0$ that solves the equations (2.26) and (2.28).

Given (R_0, θ) we can use the previous analysis to construct all the other equilibrium objects. Let $R_t = R_0 e^{g_z t}$, with $g_z = g_A/\alpha$. Let $S_t(z)$ be given by (2.22). Let $G_0(z)$ be given by (2.30) and $G_t(ze^{g_z t}) = G_0(z)$. Moreover, let u, h_{UE} and h_{EU} be given by (2.31), (2.33) and (2.34). By construction R_0 satisfies the equilibrium condition (2.5) for $t \geq 0$. By construction, θ satisfies the equilibrium condition (2.10) for all $t \geq 0$. The equilibrium conditions (2.11)-(2.13) for the UE, EU and unemployment rates hold for all $t \geq 0$. The distribution G_t satisfies the inflow-outflow condition (2.15) at all dates $t \geq 0$. Therefore, the above objects constitute a BGP. There can exist no other BGP because there is a unique solution (R_0, θ) to (2.26) and (2.28).

We have established the following result.

Proposition 2 *(Existence and Uniqueness of BGP) Consider arbitrary growth rates $g_y > 0$ and $g_A > 0$ for the production technology and the search technology. A BGP exists if and only if F is a Pareto distribution with coefficient $\alpha > 1$, the growth rates of unemployment income and vacancy costs are $g_b = g_k = g_y + g_A/\alpha$, and the discount rate is $r > g_y + g_A/\alpha$. If the BGP exists, it is unique and such that:*

(i) the unemployment rate u, the labor market tightness θ, the UE and EU rates are constant;

(ii) the initial distribution G_0 of employed workers is Pareto truncated on the left at R_0;

(iii) the reservation quality R_t and the distribution G_t grow at the rate g_A/α;

(iv) labor productivity and aggregate output grow at the rate $g_y + g_A/\alpha$.

Proposition 2 proves that a BGP exists if and only if the distribution of quality of new matches is Pareto with some coefficient α, and the worker's income from unemployment and the firm's cost from opening a vacancy grow at the same rate as the economy. In the BGP, the unemployment rate, the UE rate, the EU rate, the tightness of the labor market and the vacancies are all constant over time, even though the search technology constantly improves. The growth rate of the economy, as measured by the growth rate of labor productivity or aggregate output, is $g_y + g_A/\alpha$, i.e. the sum of the growth rate in the production technology and the growth rate in the search technology scaled by the tail coefficient of the Pareto distribution of the quality of new matches.

The intuition behind Proposition 2 is simple. Improvements in the search technology have two countervailing effects on the UE rate. On the one hand, improvements in the search technology increase the rate at which unemployed workers meet vacancies. Specifically, the meeting rate grows at the rate g_A. On the other hand, improvements in the search technology make workers and firms more selective with respect to the quality of the matches that are created, as workers can more easily explore alternative options. In particular, the expected surplus of a meeting (divided by the common component of producivity y_t) grows at the rate $-g_A(\alpha-1)/\alpha$. The reservation quality of a match, which is proportional to the product of the meeting rate and the expected surplus, grows at the rate g_A/α. And since the distribution of match qualities is Pareto with coefficient α, the probability that a meeting has an acceptable quality falls at the rate g_A. The two effects engendered by improvements in the search technology exactly cancel out and the UE rate remains constant. The EU rate is constant because the reservation quality grows at a constant rate. And if both the UE and EU rates are constant, so is unemployment.

The tightness of the labor market remains constant over time, as the cost and benefit of opening a vacancy grow at the same rate. The benefit of opening a vacancy, which is proportional to the product between the rate at which a vacancy meets an unemployed worker and the expected surplus of a match, grows at the rate $g_y + g_A/\alpha$. The cost grows, by assumption, at the same rate. Thus, improvements in the search technology have no effect on the tightness of the labor market either. And, since both unemployment and tightness are constant, so are vacancies.

Even though improvements in the search technology do not have any effect on unemployment, vacancies, UE and EU rates, they do affect the economy as they allow firms and workers to become more and more selective. Consider, for instance, the average productivity of new matches, which is given by

$$\frac{1}{1-F(R_t)}\int_{R_t} y_t z F'(z)dz = \frac{\alpha}{\alpha-1} y_t R_t. \tag{2.35}$$

The growth rate of the productivity of new matches is the sum of the growth rate g_y of the production technology and the growth rate g_A of the search technology divided by the coefficient

α of the Pareto distribution of new matches. Clearly, as old matches are drawn from the same Pareto distribution F and survive only if $z \geq R_t$, the average productivity of all matches is equal to the average productivity of new matches and, hence, grows at the rate $g_y + g_A/\alpha$.

3 Search on the job

In this section, we extend the analysis of the necessary and sufficient conditions for the existence of a BGP to a version of the model in which workers search the labor market not only when they are unemployed but also when they already have a job. The extension is relevant for the purpose of empirical realism, as it is well-known that the rate at which workers transit from one employer directly to another is around 2% per month in the US, which is nearly as large as the rate at which workers transit from employment to unemployment (see, e.g., Menzio and Shi 2011). More importantly, the extension is a fundamental robustness check for our theory of the lack of transmission on unemployment and vacancies of advancements in the search technology. In fact, the option of searching on the job affects how picky an unemployed worker is when deciding to accept or reject a job of a given quality. And, as it is clear from the previous section, the dynamics of the reservation quality are the key behind our theory of the long-run stability of the unemployment rate. Similarly, the presence of employed workers in the pool of searchers affects a firm's return from opening a vacancy. And, as it is clear from the previous section, the dynamics of the return from opening a vacancy are critical to establish the long-run stability of the vacancy rate. Reassuringly, we find that—even though the analysis of the model with search off and on the job is a good deal harder—the conditions for the existence of a BGP remain essentially the same as in Proposition 2. We conclude the section by illustrating a simple strategy to identify, empirically, the contribution to aggregate growth of improvements in the search technology.

3.1 Environment

We modify the environment of Section 2 to allow workers to search both off and on the job. In particular, we assume that unemployed workers search the labor market with an intensity of 1, and employed workers search the labor market with an intensity of $\rho \in [0,1]$. The outcome of the search process is a flow $A_t M(s_t, v_t)$ of bilateral meetings between workers and vacancies, where $s_t = u_t + \rho(1-u_t)$ is the search activity of the workers and v_t are the vacancies opened by firms. The outcome of the search process implies that an unemployed worker meets a vacancy at the rate $A_t p(\theta_t)$, where $\theta_t = v_t/s_t$ denotes the tightness of the labor market and $p(\theta) = M(1,\theta)$. Similarly, an employed worker meets a vacancy at the rate $\rho A_t p(\theta_t)$. A vacancy meets a worker at the rate $A_t q(\theta_t)$, where $q(\theta) = p(\theta)/\theta$.

When a firm and a worker meet, they observe the quality $\hat{z}$ of their match, which is drawn from the distribution F. Based on this information, the firm and the worker decide whether to match or not. If they decide not to match, the worker remains in his old employment position (either unemployment or employment in a match of quality z) and the firm's vacancy remains unfilled. If they decide to match, the firm and the worker bargain over the terms of an employment contract and start producing together. We assume that the contingencies of the employment contract are rich enough so as to maximize the joint value of the match. We assume that the gains from trade are allocated according to the Axiomatic Nash bargaining solution, where the threat point of the firm is the value of an unfilled vacancy, the threat point of an unemployed worker is the value of unemployment, and the threat point of an employed worker is the joint value of the match with his old employer. The assumption that the threat point of an employed worker is the joint value of the match with his old employer is the same as in Postel-Vinay and Robin (2002) or in Bagger et al. (2014). The assumption captures the idea that the old employer is aware of and responds to the worker's outside offers.

3.2 Definition of BGP

The joint value of a firm-worker match of quality z is given by

$$\begin{aligned} V_t(z) = \max_{d \geq 0} \quad & \int_0^d e^{-rx} \mu_{t+x}(z) \left[y_{t+x} z + A_{t+x} p(\theta) \rho \gamma \int_z (S_{t+x}(\hat{z}) - S_{t+x}(z)) dF(\hat{z}) \right] dx \\ & + e^{-rd} \mu_{t+d}(z) U_{t+d}, \end{aligned} \tag{3.1}$$

where μ_{t+x} denotes the probability that the match is still active at date $t+x$ and is equal to

$$\mu_{t+x}(z) = \exp\left[-\int_0^x A_{t+s} p(\theta) \rho (1 - F(z)) ds \right]. \tag{3.2}$$

Let us explain the above expressions in some detail. The rate at which the firm-worker match is dissolved at any date $t+s$ is $A_{t+s} p(\theta)(1-F(z))$, which is the rate at which the worker contacts another firm and the quality of their match is greater than z. Conditional on the firm-worker match surviving to date $t+x$, the joint income at date $t+x$ is given by $y_{t+x} z$. Moreover, at date $t+x$, the worker contacts a new firm at the rate $A_{t+x} p(\theta)$. If the quality $\hat{z}$ of the match between the worker and the new firm is smaller than z, the match between the worker and the new firm continues. Otherwise, the worker moves to the new firm and captures a fraction γ of the gains from trade, which are given by $V_{t+x}(\hat{z}) - V_{t+x}(z) = S_{t+x}(\hat{z}) - S_{t+x}(z)$. Conditional on the firm-worker match surviving to date $t+d$, the match is dissolved and the continuation joint value to the worker and the firm is the worker's lifetime income from unemployment U_{t+d}.

The break-up date d^* maximizes the joint value of the firm-worker match if and only if it satisfies the following condition

$$y_{t+d}z+A_{t+d}p(\theta)\rho\gamma\int_{z}(S_t(\hat{z})-S_t(z))dF(\hat{z})\leq rU_{t+d}-\mathring{U}_{t+d},\text{ and } d\geq 0, \tag{3.3}$$

where the two inequalities hold with complementary slackness. The condition in (3.3) is the analogue of condition (2.4). In fact, the left-hand side of (3.3) is the marginal benefit from delaying the break-up of the match. The right-hand side of (3.3) is the marginal cost of delaying the break-up of the match.

The reservation quality R_t is defined as

$$y_tR_t+A_tp(\theta)\rho\gamma\int_{R_t}(S_t(\hat{z})-S_t(R_t))dF(\hat{z})=rU_t-\mathring{U}_t. \tag{3.4}$$

Given the above definition and the optimality condition (3.3) for the break-up date, it follows that, when a firm and an unemployed worker meet, they form a match if and only if they observe a quality $z>R_t$. Similarly, when a firm and a worker are in a match, they keep the match alive if and only if $z>R_t$.

The present value of income for an unemployed worker is such that

$$rU_t=b_t+A_tp(\theta)\gamma\int_{R_t}S_t(\hat{z})dF(\hat{z})+\mathring{U}_t. \tag{3.5}$$

The annuitized present value of income of an unemployed worker is the sum of three terms. The first is the worker's income b_t in unemployment. The second is the rate at which an unemployed worker meets a firm times a fraction γ of the gains from trade $V_t(\hat{z})-U_t=S_t(\hat{z})$ if the quality $\hat{z}$ of the match is greater than R_t. The last term is the time derivative of U_t.

From (3.1) and (3.5), it follows that the surplus $S_t(z)$ of a firm-worker match of quality $z>R_t$ is such that

$$rS_t(z)=y_tz-b_t+A_tp(\theta)\rho\gamma\int_{z}(S_t(\hat{z})-S_t(z))dF(\hat{z})-A_tp(\theta)\gamma\int_{R_t}S_t(\hat{z})dF(\hat{z})+\mathring{S}_t(z). \tag{3.6}$$

The annuitized surplus of the match is given by three terms. The first one is the difference in the income created by the firm-worker match and the income of an unemployed worker. The second term is the difference between the option value of search for the firm-worker match and the option value of search for an unemployed worker. The last term is the time derivative of the surplus.

The tightness θ of the labor market is such that

$$\begin{aligned} k_t = \quad & A_tq(\theta)\frac{u}{u+\rho(1-u)}(1-\gamma)\int_{R_t}S_t(\hat{z})dF(\hat{z}) \\ & A_tq(\theta)\frac{\rho(1-u)}{u+\rho(1-u)}(1-\gamma)\int_{R_t}\left[\int_{z}(S_t(\hat{z})-S_t(z))\,dF(\hat{z})\right]dG_t(z). \end{aligned} \tag{3.7}$$

The left-hand side is the cost to the firm of maintaining a vacancy. The right hand side is the benefit to the firm of maintaining a vacancy. This benefit is given by the sum of two terms. The first term is the product of the rate at which the vacancy contacts a worker, the probability that the contacted worker is unemployed, and a fraction $1-\gamma$ of the expected gains from trade between the firm and the worker, which are equal to $S_t(\hat{z})$ if the quality of the match between the firm and the worker is $\hat{z} > R_t$ and zero otherwise. The second term is the product of the rate at which the firm contacts a worker, the probability that the worker is employed, and a fraction $1-\gamma$ of the expected gains from trade, which are equal to $S_t(\hat{z}) - S_t(z)$ if the quality of the match between the firm and the worker is $\hat{z} > z$, where z denotes the quality of the worker's current match.

The stationarity condition for the UE, EU and unemployment rates are (2.11), (2.12) and (2.13), the same conditions as in the version of the model without search on the job. The condition that guarantees that the cross-sectional distribution G_t of employed workers across matches of different quality "grows" at the constant rate g_z—in the sense that every quantile xth of the distribution G_t grows at the rate g_z—is

$$\begin{aligned} & (1-u)G_t'(z_t(x))z_t(x)g_z + uA_t p(\theta)\left[F(z_t(x)) - F(R_t)\right] \\ = \;& (1-u)G_t'(R_t)R_t g_z + (1-u)\rho A_t p(\theta)[1-F(z_t(x))]G_t(z_t(x)). \end{aligned} \tag{3.8}$$

The left-hand side is the flow of workers who are not among those employed in matches with a quality below the xth quantile at date t and become employed in such matches over the next instant. The right-hand side is the flow of workers who are among those employed in matches with a quality below the xth quantile at date t and leave such matches over the next instant. The only difference between (3.8) and its analogue (2.15) is the second term on the right-hand side which represents workers who are employed in a match below the x quantile and, through search on the job, move to a match above the xth quantile.

The above observations motivate the following definition of a BGP.

Definition 2: *A BGP is a tuple $\{R_t, S_t, U_t, \theta, h_{UE}, h_{EU}, u, G_t\}$ such that for all $t \geq 0$: (i) R_t, U_t and S_t satisfy (3.4), (3.5) and (3.6); (ii) θ satisfies (3.7); (iii) h_{UE}, h_{EU} and u satisfy (2.11), (2.12) and (2.13); (iv) G_t satisfies (3.8).*

3.3 Existence of a BGP

The version of the model in which workers only search off the job is a special case of the version of the model in which workers may search off and on the job. Therefore, the conditions on fundamentals that are necessary for the existence of a BGP in the version of the model with off-the-job search, listed in Proposition 1, are also necessary for the existence of a BGP in the more general model. That is, a BGP exists only if the distribution F of match qualities is Pareto with coefficient α, the growth rate g_b of the worker's unemployment income and the growth rate g_k of the firm's vacancy cost are equal to $g_y + g_A/\alpha$, and the discount rate r is greater than $g_y + g_A/\alpha$. Moreover, in any BGP, the "growth rate" g_z of the cross-sectional distribution G_t of employed workers across different matches must be equal to g_A/α.

Given the above assumptions, we can now construct a candidate BGP. The reservation quality R_t satisfies condition (3.4). Using the equilibrium condition (3.5) to replace $rU_t - \mathring{U}_t$, we can rewrite (3.4) as

$$\begin{aligned} & y_t R_t + A_t p(\theta)\rho\gamma \textstyle\int_{R_t} (S_t(\hat{z}) - S_t(R_t)) dF(\hat{z}) \\ = \quad & b_t + A_t p(\theta)\gamma \textstyle\int_{R_t} S_t(\hat{z}) dF(\hat{z}). \end{aligned} \tag{3.9}$$

The surplus $S_t(z)$ satisfies the equilibrium condition (3.6). Using (3.9), we can rewrite (3.6) as

$$rS_t(z) = y_t(z - R_t) - A_t p(\theta)\rho\gamma \left[S_t(z)(1 - F(z)) + \int_{R_t}^{z} S_t(\hat{z}) dF(\hat{z}) \right] + \mathring{S}_t(z). \tag{3.10}$$

Intuitively, (3.10) says that the annuitized surplus of a firm-worker match is the difference between the flow income it generates and the flow income generated by a match of quality R_t plus the difference between the option value of on-the-job search for the firm-worker match and the option value of on-the-job search for a match of quality R_t.

We guess that the surplus function S_t that solves the Bellman Equation (3.10) is such that

$$S_t(ze^{g_z t}) = S_0(z) e^{(g_y + g_z)t}, \; \forall z \geq R_0, \; \forall t \geq 0, \tag{3.11}$$

where $S_0(z)$ is given by

$$\begin{aligned} rS_0(z) \quad & = y_0(z - R_0) + S_0(z) \left[g_y + g_z - A_0 p(\theta)\rho\gamma \left(\frac{z_\ell}{z} \right)^\alpha \right] \\ & -A_0 p(\theta)\rho\gamma\alpha \textstyle\int_{R_0}^{z} S_0(\hat{z}) \left(\frac{z_\ell}{\hat{z}} \right)^\alpha \frac{1}{\hat{z}} d\hat{z} - S_0'(z) z g_z. \end{aligned} \tag{3.12}$$

In words, we guess that, when evaluated at a match quality that grows at the constant rate g_z, the surplus function S_t grows at the constant rate $g_y + g_z$.

To verify that the guess is correct, we need to check that the proposed solution does satisfy the Bellman Equation (3.10) for all z and all t. Consider the Bellman Equation (3.10) for $S_t(ze^{g_z t})$. Plugging the proposed solution $S_t(ze^{g_z t}) = S_0(z)e^{(g_y+g_z)t}$ into the equation and using the fact that F is a Pareto with coefficient α gives

$$\begin{aligned} rS_0(z)e^{(g_y+g_z)t} \quad & = y_t(ze^{g_z t} - R_t) - A_t p(\theta)\rho\gamma S_0(z) e^{(g_y+g_z)t} \left(\frac{z_\ell}{ze^{g_z t}} \right)^\alpha \\ & -A_t p(\theta)\rho\gamma\alpha \int_{R_t}^{ze^{g_z t}} S_t(\hat{z}) \left(\frac{z_\ell}{\hat{z}} \right)^\alpha \tfrac{1}{\hat{z}} d\hat{z} + \mathring{S}_t(ze^{g_z t}). \end{aligned} \tag{3.13}$$

Notice that $S_t(ze^{g_z t}) = S_0(z)e^{(g_y+g_z)t}$ implies

$$\begin{aligned}
\mathring{S}_t(ze^{g_z t}) &= \lim_{dt\to 0}\frac{1}{dt}\left[S_{t+dt}(ze^{g_z t}) - S_t(ze^{g_z t})\right] \\
&= \lim_{dt\to 0}\frac{1}{dt}\left[S_0(z)e^{(g_y+g_z)(t+dt)} - S_0(z)e^{(g_y+g_z)t}\right]/dt \\
&- \lim_{dt\to 0}\frac{1}{dt}\left[S_0(z)e^{(g_y+g_z)(t+dt)} - S_0(ze^{-g_z dt})e^{(g_y+g_z)(t+dt)}\right]/dt \\
&= (g_y+g_z)S_0(z)e^{(g_y+g_z)t} - zg_z S_0'(z)e^{(g_y+g_z)t}.
\end{aligned} \tag{3.14}$$

Further, from $S_t(ze^{g_z t}) = S_0(z)e^{(g_y+g_z)t}$, $R_t = R_0 e^{g_z t}$ and making the change of variable $\hat{x} = \hat{z}e^{-g_z t}$, we obtain

$$\int_{R_t}^{ze^{g_z t}} S_t(\hat{z})\left(\frac{z_\ell}{\hat{z}}\right)^\alpha \frac{1}{\hat{z}}d\hat{z} = \int_{R_0}^{z} S_0(\hat{x})e^{(g_y+g_z)t}\left(\frac{z_\ell}{\hat{x}}\right)^\alpha \frac{1}{\hat{x}}e^{-\alpha g_z t}d\hat{x}. \tag{3.15}$$

From (3.14)-(3.15), $y_t = y_0 e^{g_y t}$, $A_t = A_0 e^{g_A t}$, $R_t = R_0 e^{g_z t}$ and $g_z = g_A/\alpha$, it follows that the Bellman Equation (3.13) can be written as

$$\begin{aligned}
& rS_0(z)e^{(g_y+g_z)t} \\
= \;& e^{(g_y+g_z)t}\left\{y_0(z-R_0) - A_0 p(\theta)\rho\gamma S_0(z)\left(\frac{z_\ell}{z}\right)^\alpha \right. \\
& \left. -A_0 p(\theta)\rho\gamma\alpha\int_{R_0}^{z} S_0(\hat{x})\left(\frac{z_\ell}{\hat{x}}\right)^\alpha \frac{1}{\hat{x}}d\hat{x} + (g_y+g_z)S_0(z) - zg_z S_0'(z)\right\}
\end{aligned} \tag{3.16}$$

The above equation is satisfied given our guess for S_0 in (3.12). Therefore, we have verified that the surplus function given by (3.12) and $S_t(ze^{g_z t}) = S_0(z)e^{(g_y+g_z)t}$ is indeed the solution to the Bellman Equation (3.10).

To solve for the surplus function $S_0(z)$, we differentiate (3.12) with respect to z and obtain

$$rS_0'(z) = y_0 + S_0'(z)\left[g_y - A_0 p(\theta)\rho\gamma\left(\frac{z_\ell}{z}\right)^\alpha\right] - zg_z S_0''(z). \tag{3.17}$$

The equation above is a differential equation for the derivative of the surplus function $S_0(z)$ with respect to z. The solution to the differential equation which satisfies the smooth-pasting condition $S_0'(R_0) = 0$ is

$$S_0'(z) = \frac{y_0}{g_z}\int_{R_0}^{z}\frac{1}{s}\exp\left\{-\frac{1}{g_z}\left[\frac{\sigma}{\alpha}(F(z)-F(s)) + (r-g_y)\log(z/s)\right]\right\}ds, \tag{3.18}$$

where $\sigma = A_0 p(\theta)\rho\gamma$. Then, using the fact that $S_0(z) = S_0(R_0) + \int_{R_0}^{z} S_0'(x)dx$ and $S_0(R_0) = 0$, we find that the surplus function $S_0(z)$ is

$$S_0(z) = \frac{y_0}{g_z}\int_{R_0}^{z}\left[\int_{R_0}^{x}\frac{1}{s}\exp\left\{-\frac{1}{g_z}\left[\frac{\sigma}{\alpha}(F(x)-F(s)) + (r-g_y)\log(z/s)\right]\right\}ds\right]dx. \tag{3.19}$$

The expected gains from trade between a firm and an unemployed worker, which we denote as $S^e_{u,t}$, are given by

$$\begin{aligned} S^e_{u,t} &= \alpha \int_{R_t} S_t(\hat{z}) \left(\frac{z_\ell}{\hat{z}}\right)^\alpha \frac{1}{\hat{z}} d\hat{z} \\ &= \alpha e^{(g_y-(\alpha-1)g_z)t} \int_{R_0} S_0(\hat{x}) \left(\frac{z_\ell}{\hat{x}}\right)^\alpha \frac{1}{\hat{x}} d\hat{x} \\ &= e^{(g_y-(\alpha-1)g_z)t} S^e_{u,0}. \end{aligned} \tag{3.20}$$

Similarly, the expected gains from trade between a firm and worker employed in a job of quality $ze^{g_z t}$, which we denote as $S^e_{e,t}(ze^{g_z t})$ are given by

$$\begin{aligned} S^e_{e,t}(ze^{g_z t}) &= \alpha \int_{ze^{g_z t}} \left(S_t(\hat{z}) - S_t(ze^{g_z t})\right) \left(\frac{z_\ell}{\hat{z}}\right)^\alpha \frac{1}{\hat{z}} d\hat{z} \\ &= \alpha e^{(g_y-(\alpha-1)g_z)t} \int_{z} \left(S_0(\hat{x}) - S_0(z)\right) \left(\frac{z_\ell}{\hat{x}}\right)^\alpha \frac{1}{\hat{x}} d\hat{x} \\ &= e^{(g_y-(\alpha-1)g_z)t} S^e_{e,0}(z). \end{aligned} \tag{3.21}$$

Finally, the expected gains from trade between a firm and an employed worker are given by

$$\int_{R_t} S^e_{e,t}(z) G'_t(z) dz = e^{(g_y-(\alpha-1)g_z)t} \int_{R_0} S^e_{e,0}(x) G'_0(x) dx. \tag{3.22}$$

The expressions in (3.20) and (3.21) are obtained by making use of the fact that $S_t(ze^{g_z t}) = S_0(z)e^{(g_y+g_z)t}$ and $R_t = R_0 e^{g_z t}$, and then by changing the variable of integration from $\hat{z}$ to $\hat{x} = \hat{z}e^{-g_z t}$. The expression in (3.22) is obtained by making use of the fact that $S^e_{e,t}(ze^{g_z t}) = e^{(g_y-(\alpha-1)g_z)t} S^e_{e,0}(z)$ and $G'_t(ze^{g_z t}) = G'_0(z)e^{-g_z t}$, which follows from $G_t(ze^{g_z t}) = G_0(z)$, and then by changing the variable of integration from z to $x = ze^{-g_z t}$. It is easy but tedious to verify that the integrals in (3.20)-(3.22) are finite if and only if $\alpha > 1$.

The reservation quality R_t is given by the equilibrium condition (3.9). Using the fact that $S_t(R_t) = 0$, condition (3.9) can be written as

$$R_t = \frac{b_t}{y_t} + \frac{A_t}{y_t} p(\theta)(1-\rho)\gamma e^{(g_y-(\alpha-1)g_z)t} S^e_{u,0}. \tag{3.23}$$

When evaluated at date $t = 0$, (3.23) holds if and only if the reservation quality R_0 is given by

$$R_0 = \frac{b_0}{y_0} + \frac{A_0}{y_0} p(\theta)(1-\rho)\gamma S^e_{u,0}. \tag{3.24}$$

When evaluated at date $t \geq 0$, (3.23) holds if and only if the growth rate of the left-hand side is equal to the growth rate of each of the two terms on the right-hand side of (3.23). The growth rate of the left-hand side is g_z, the growth rate of the reservation quality. The growth rate of the first term on the right-hand side is g_A/α and the growth rate of the second term is $g_A - (\alpha-1)g_z$.

Clearly, these growth rates are equal if and only if the growth rate g_z of the reservation quality is equal to g_A/α.

The tightness of the labor market θ is given by the equilibrium condition (3.7). Using (3.20) and (3.22), condition (3.7) can be written as

$$\begin{aligned} k_t = \; & A_t q(\theta)(1-\gamma)e^{(g_y-(\alpha-1)g_z)t} \\ & \times \left\{ \frac{u}{u+\rho(1-u)} S^e_{u,0} + \frac{\rho(1-u)}{u+\rho(1-u)} \int_{R_0} S^e_{e,0}(x) G_0'(x)dx \right\} \end{aligned} \tag{3.25}$$

When evaluated at date $t = 0$, (3.25) holds if and only if θ satisfies

$$\begin{aligned} k_0 = \; & A_0 q(\theta)(1-\gamma) \\ & \times \left\{ \frac{u}{u+\rho(1-u)} S^e_{u,0} + \frac{\rho(1-u)}{u+\rho(1-u)} \int_{R_0} S^e_{e,0}(x) G_0'(x)dx \right\} \; . \end{aligned} \tag{3.26}$$

When evaluated at $t \geq 0$, (3.25) holds if and only if the growth rate of the left-hand side, which is g_k, equals the growth rate of the right-hand side, which is $g_y - (\alpha-1)g_z + g_A$. Since $g_k = g_y + g_A/\alpha$ and $g_z = g_A/\alpha$, the growth rates of the left and right-hand sides are the same.

The distribution $G_t(ze^{g_z t}) = G_0(z)$ of employed workers across matches of different qualities must satisfy the inflow-outflow condition (3.8). Since the stationarity condition (2.13) for unemployment implies that the flow of workers into unemployment $(1-u)G_t'(R_t)R_t g_z$ is equal to the flow out $uA_t p(\theta)(1-F(R_t))$, we can rewrite (3.8) as

$$\begin{aligned} & (1-u)G_t'(ze^{g_z t})ze^{g_z t}g_z \\ = \; & uA_t p(\theta)\left(\frac{z_\ell}{ze^{g_z t}}\right)^\alpha + (1-u)A_t p(\theta)\rho\left(\frac{z_\ell}{ze^{g_z t}}\right)^\alpha G_t(ze^{g_z t}). \end{aligned} \tag{3.27}$$

We first use the expression in (3.27) to recover the initial distribution G_0 of employed workers across matches of different qualities. To this aim, note that (3.27) at $t = 0$ is a differential equation for G_0. The solution to the differential equation that satisfies the boundary condition $G_0(\infty) = 1$ is

$$G_0(z) = \left(1 + \frac{u}{\rho(1-u)}\right)\exp\left(-A_0 p(\theta)\rho(1-F(z))/g_A\right) - \frac{u}{\rho(1-u)}. \tag{3.28}$$

Since $G_0(R_0)$ must be equal to 0, (3.28) implies that the unemployment rate u must be

$$u = \frac{\rho\exp\left(-A_0 p(\theta)\rho(1-F(R_0))/g_A\right)}{1-(1-\rho)\exp\left(-A_0 p(\theta)\rho(1-F(R_0))/g_A\right)}. \tag{3.29}$$

Using (3.29), we can rewrite (3.28) as

$$G_0(z) = \frac{\exp\left(-A_0 p(\theta)\rho(1-F(z))/g_A\right) - \exp\left(-A_0 p(\theta)\rho(1-F(R_0))/g_A\right)}{1-\exp\left(-A_0 p(\theta)\rho(1-F(R_0))/g_A\right)}. \tag{3.30}$$

That is, the initial distribution G_0 of employed workers across matches of different qualities is a Fréchet distribution truncated at the reservation quality R_0.

We then need to verify that the balanced growth distribution $G_t(ze^{g_z t}) = G_0(z)$ satisfies (3.27) for all $t \geq 0$. This is the case if and only if G_0 is given by (3.30) and the left-hand side of (3.27) grows at the same rate as the right-hand side of (3.27). The left-hand side grows at the rate of 0, as $G_t'(ze^{g_z t}) = G_0'(z)e^{-g_z t}$. The first term on the right-hand side grows at the rate of 0, as $A_t = A_0 e^{g_A t}$ and $g_z = g_A/\alpha$. The second term on the right-hand side grows at the rate of 0 as well, as $G_t(ze^{g_z t}) = G_0(z)$. Therefore, as long as G_0 is given by (3.30), (3.27) holds for $t = 0$ and, consequently, for all $t \geq 0$.

Finally, we need to solve for the UE and EU rates. The conditions (2.11) and (2.12) for the UE and EU rates at date $t = 0$ can be written as

$$h_{UE} = A_0 p(\theta) \left(\frac{z_\ell}{R_0}\right)^\alpha, \tag{3.31}$$

$$\begin{aligned} h_{EU} &= G_0'(R_0) R_0 g_z \\ &= A_0 p(\theta) \left(\frac{z_\ell}{R_0}\right)^\alpha \frac{\exp\left(-A_0 p(\theta)\rho(1-F(R_0))/g_A\right)}{1-\exp\left(-A_0 p(\theta)\rho(1-F(R_0))/g_A\right)} \end{aligned} \tag{3.32}$$

The above value for h_{UE} satisfies condition (2.11) for all $t \geq 0$, as A_t grows at the rate g_A and $(z_\ell/R_t)^\alpha$ grows at the rate $-\alpha g_z = -g_A$. The above value for h_{EU} satisfies condition (2.12) for all $t \geq 0$, as $G_t'(R_t)$ grows at the rate $-g_z$ and R_t grows at the rate g_z.

We are now in the position to prove the existence of a BGP. Any BGP must be such that R_0 and θ satisfy the date $t = 0$ conditions (3.24) and (3.26). Given an R_0 and θ that satisfy (3.24) and (3.26), we construct all the other equilibrium objects. In particular, the reservation quality R_t is given by $R_0 e^{g_z t}$ with $g_z = g_A/\alpha$. The surplus function $S_t(ze^{g_z t})$ is given by $S_0(z)e^{(g_y+g_z)t}$, where $S_0(z)$ is as in (3.19). The unemployment rate u, the UE rate h_{UE} and the EU rate h_{EU} are given by (3.29), (3.31) and (3.32). The distribution $G_t(ze^{g_z t})$ of employed workers is given by $G_0(z)$, where $G_0(z)$ is as in (3.30). Clearly, these objects are the only ones satisfying the conditions for a BGP. As in Section 2, it is easy to show that there exists at least one solution (R_0, θ) to the conditions (3.24) and (3.26). However, we are not able to establish its uniqueness.

We summarize our findings in the following proposition.

Proposition 3 *(Existence of BGP with On-the-Job Search) Consider arbitrary growth rates $g_y > 0$ and $g_A > 0$ for the production technology and the search technology. A BGP exists if and only if F is a Pareto distribution with coefficient $\alpha > 1$, the growth rates of unemployment income and vacancy costs are $g_b = g_k = g_y + g_A/\alpha$, and the discount rate is $r > g_y + g_A/\alpha$. In any BGP:*

(i) the unemployment rate u, the labor market tightness θ, the UE and EU rates are constant;

(ii) the initial distribution G_0 of employed workers is Fréchet truncated on the left at R_0;

(iii) the reservation quality R_t and the distribution G_t grow at the rate g_A/α;

(iv) labor productivity and aggregate output grow at the rate $g_y + g_A/\alpha$.

3.4 Properties of the BGP and identification

A BGP for the model in which workers search off and on the job exists under essentially the same conditions as for the simpler model of Section 2. In a BGP, the unemployment rate, the UE rate, the EU rate, the EE rate, the tightness of the labor market and vacancies are all constant in the face of an ever improving efficiency of the search technology.

It is worthwhile to briefly discuss the intuition behind the independence of unemployment and the other workers' transition rates and the efficiency of the search technology. The rate at which unemployed workers meet firms grows at the rate g_A, the growth rate of the search technology. The expected surplus of a match between a firm and an unemployed worker grows at the rate $g_y - (\alpha - 1)g_z = g_y - (\alpha - 1)g_A/\alpha$, where g_y is the growth rate of the production technology and α is the coefficient of the Pareto distribution for the quality of a new match. The reservation quality, which is proportional to the product of the meeting rate and the expected surplus divided by the common component of productivity, increases at the rate g_A/α. Since the quality of new matches is drawn from a Pareto with coefficient α, the probability that a meeting between a firm and a worker turns into a match grows at the rate $-g_A$ and the UE rate remains constant over time. Note that the growth rate of the reservation quality is the same as in the model without on-the-job search. The option of searching on the job lowers the reservation quality but it does not affect its growth rate, because the on-the-job search option ρ only lowers the constant of proportionality on the expected surplus in the reservation quality equation.

Once workers move from unemployment into employment, they keep on searching. On the one hand, the rate at which newly employed workers meet vacancies increases over time at the rate g_A. On the other hand, the distribution of match qualities for newly employed workers increases at the rate g_A/α because of the increase in the reservation quality. Since the quality of new matches is drawn from a Pareto with coefficient α, the rate at which employed workers find better jobs is constant over time. The overall result of the process through which workers move from unemployment into employment and from lower to higher quality jobs is a cross-sectional distribution with a stable shape (a truncated Fréchet) and a constant growth rate of g_A/α. Since the distribution grows at the same rate as the reservation quality, the EU rate is constant over time. In turn, since the UE and EU rates are constant, so is unemployment.

Now, consider the labor market from the perspective of firms. The benefit of opening a vacancy grows at a constant rate, given by the sum of the growth rate g_A of the search technology and the growth rate $-(\alpha-1)g_A/\alpha$ of the expected value to the firm from a meeting a worker. The sum of these two growth rates is the overall growth rate of the economy. Assuming that the cost of opening a vacancy also grows at the same rate as the economy, the tightness of the labor market remains constant over time. In turn, as the measure of searching workers and the tightness of the labor market are constant, so is the number of vacancies. Note that the growth rate of the expected value to the firm from meeting a worker is the same as in the model without on-the-job search. Intuitively, the option of searching on the job implies that the firm will meet both unemployed and employed workers. The employment status of whom the firm meets affects the expected value of the meeting to the firm, as unemployed workers are more likely to match with the firm and, if they do, they capture a smaller share of the surplus than employed workers. However, the composition of workers is stationary over time, and it does not affect the growth rate of the firm's expected benefit from a meeting.

The growth rate of the search technology contributes to the overall growth of the economy. In fact, note that the average productivity of labor is given by

$$\int_{R_t} y_t z G_t'(z)dz = e^{(g_y+g_A/\alpha)t}\int_{R_0} y_0 G_0'(z)dz. \tag{3.33}$$

The above expression shows that the growth rate of the average productivity of labor is the sum of the growth rate g_y of the production technology and the ratio between the growth rate g_A of the search technology and the coefficient α of the Pareto distribution for the quality of new matches. Since employment is constant over time, the growth rate of aggregate output is also given by g_y+g_A/α. Intuitively, improvements in the search technology contribute to the growth rate of the economy because they allow firms and workers to become more selective and create higher-quality matches. The magnitude of the contribution of improvements in the search technology on the growth rate of the economy depends on the shape of the Pareto distribution of the quality of new matches. In particular, the thicker is tail if the distribution (i.e. the lower is α), the larger is the contribution of improvements in the search technology. Intuitively, the thicker is the tail of the Pareto distribution the higher the rate of return on improvements in search. Note that the growth rate of the economy is the same as in the model without on-the-job search. Intuitively, this is the case because the option of searching on the job improves the distribution of workers across matches of different quality, but not the rate at which the distribution grows.

Naturally, it would be interesting to measure the contribution g_A/α of improvements in the search technology to the growth rate of the economy. To this aim, note that the growth rate g_A of the search technology can be measured by looking at how more selective firms become over time. The number of workers considered by a firm per unit of time is given by

$$A_t q(\theta) = A_0 q(\theta) e^{g_A t}. \tag{1}$$

The rate at which the firm fills its vacancy is given by

$$\begin{aligned} & A_t q(\theta) \left\{ \frac{u}{u+\rho(1-u)} [1-F(R_t)] + \frac{\rho(1-u)}{u+\rho(1-u)} \int_{R_t} [1-F(z)] G_t'(z) dz \right\} \\ = \quad & A_0 q(\theta) \left\{ \frac{u}{u+\rho(1-u)} [1-F(R_0)] + \frac{\rho(1-u)}{u+\rho(1-u)} \int_{R_0} [1-F(z)] G_0'(z) dz \right\}. \end{aligned} \tag{3.35}$$

The average number of workers considered for each vacancy is the product between the number of workers considered in each unit of time, (1), and the average duration of the vacancy, which is the inverse of (3.35). That is, the average number of workers considered for each vacancy is

$$e^{g_A t} \left\{ \frac{u}{u+\rho(1-u)} [1-F(R_0)] + \frac{\rho(1-u)}{u+\rho(1-u)} \int_{R_0} [1-F(z)] G_0'(z) dz \right\}^{-1}. \tag{3.36}$$

The above expression implies that the average number of workers considered for each vacancy grows at the rate g_A. Intuitively, the average number of workers applying to a vacancy per unit of time grows at the rate g_A because of declining search frictions. The average duration of a vacancy remains constant over time, because the decline in search frictions is exactly offset by the increase in selectivity. Hence, the average number of workers considered for each vacancy grows at the rate g_A.

Next, note that the coefficient α of the Pareto distribution of qualities for new matches can be measured using the cross-section of wages. To see why this is the case, it is useful to start by considering the version of the model in which workers can only search off the job. In this version of the model, the distribution G_t of employed workers across matches of different quality is Pareto with coefficient α and, if the wage is linear in the quality of the match, α can be recovered as the tail coefficient of the cross-sectional distribution of wages. Suppose, for instance, that the wage $w_t(z)$ for a worker in a match of quality z at date t is given by

$$w_t(z) = \gamma y_t z + (1-\gamma) y_t R_t. \tag{3.37}$$

The wage (3.37) is consistent with the fact that the equilibrium contract assigns to the worker a fraction γ of the surplus of the match. Given the wage (3.37), the cross-sectional distribution of wages $L_t(w)$ is given by

$$L_t(w) = G_t\left(\frac{1}{\gamma} \frac{w}{y_t} - \frac{1-\gamma}{\gamma} R_t \right) = 1 - \left(\frac{\gamma y_t R_t}{w - (1-\gamma) y_t R_t} \right)^{\alpha}. \tag{3.38}$$

The cross-sectional wage distribution $L_t(w)$ is not Pareto. However, the right tail of the wage distribution $L_t(w)$ is well approximated by a Pareto distribution with coefficient α, as

$$\lim_{w\to\infty} \frac{d\log[1-L_t(w)]}{d\log w} = -\alpha. \tag{3.39}$$

In the version of the model in which workers search both off and on the job, the mapping between the cross-sectional wage distribution and the coefficient α is more complicated. First, the distribution G_t of employed workers across matches of different quality is not Pareto with coefficient α, but Fréchet. Second, the fraction of the surplus of a match captured by a worker is not always γ, as it depends on the worker's outside option when he was hired. Workers hired out of unemployment capture a fraction γ of the surplus. Workers hired out of matches of a slightly lower quality capture almost all of the surplus. Nevertheless, the tail coefficient of the cross-sectional distribution of wages for workers hired directly out of unemployment is still α. In fact, the distribution of match qualities for workers hired out of unemployment is Pareto with coefficient α and the fraction of the surplus accruing to these workers is γ. Moreover, for $z\to\infty$, the wage $\gamma y_t z$ assigns a fraction γ of the surplus to the workers, as the option value of searching on the job is approximately 0 and the reservation quality R_t is negligible relative to z. Hence, the tail coefficient of the cross-sectional wage distribution of workers hired out of unemployment is α.

We summarize the above insights in the following proposition.

Proposition 4 *(Search Technology and Growth) The contribution of the growth rate of the search technology to the growth rate of the economy is the ratio between the growth rate of applications per vacancy and the tail coefficient of the wage distribution for workers hired out of unemployment.*

4 Population growth

In this section, we extend the analysis of necessary and sufficient conditions for a BGP to a version of the model in which the labor force grows over time rather than being constant, and the search process displays arbitrary returns to scale, rather than constant returns to scale. We find that the same conditions under which unemployment, UE, EU rates, tightness and vacancies are constant in the face of technological improvements in the search technology also guarantee that these variables remain constant in the face of growth in the labor force and non-constant returns to scale in the search process. This finding is intuitive, as both technological improvements and return to scale have the same effect of changing the rate at which workers and firms meet. The finding is important, as it means that returns to scale in search cannot be detected by regressing unemployment (or other variables, such as the UE rate) on the size of the labor force either in the time-series or in the cross-section. We conclude by developing a strategy to identify, empirically, the returns to scale in the search process and their contribution to the growth of the economy.

4.1 Environment and definition of BGP

We modify the environment of Section 2 to allow for population growth and for arbitrary returns to scales in the search technology. At date t, the labor market is populated by a continuum of workers with measure N_t, where $N_t = N_0 e^{g_N t}$ and $g_N \geq 0$ is the constant growth rate of the workforce. The flow $g_N N_t$ of workers entering the market at date t are unemployed. At date t, the labor market is also populated by a continuum of firms with some positive measure.

Workers and firms search the labor market for trading opportunities. As in Section 2, we assume that workers search the market only when unemployed. The measure of unemployed workers at date t is given by $N_t u_t$, where u_t is the unemployment rate. Firms search the market by posting vacancies at the flow cost k_t. The measure of vacancies at date t is given by $N_t v_t$, where v_t is the measure of vacancies per worker. The outcome of the search process is a flow $A_t N_t^{1+\beta} M(u_t, v_t)$ of bilateral meetings between unemployed workers and vacancies, where $A_t = A_0 e^{g_A t}$ is the efficiency of the search technology, β is a parameter that captures the scale effects in the search technology, and $M(u,v)$ is a constant returns to scale function. If $\beta = 0$, the flow of meetings $A_t N_t^{1+\beta} M(u_t, v_t)$ is linear in N_t and, hence, the search process has constant returns to scale. If $\beta > 0$, the flow of meetings is more than linear in N_t and, hence, the search process has increasing returns to scale. And, if $\beta < 0$, the flow of meetings is less than linear in N_t and, hence, the search process has decreasing returns to scale. The outcome of the search process implies that an unemployed worker meets a vacancy at the rate $\hat{A}_t p(\theta)$, where $\theta = v_t/u_t$ is the tightness of the labor market, $p(\theta) = M(1, \theta)$, and $\hat{A}_t = A_t N_t^{\beta}$. Similarly, a vacancy meets an unemployed worker at the rate $\hat{A}_t q(\theta)$, where $q(\theta) = p(\theta)/\theta$.

The remainder of the environment is the same as in Section 2. Namely, upon meeting, a firm and a worker observe the idiosyncratic component z of productivity of their match, where z is drawn from the distribution F. After observing z, the firm and the worker decide whether to form a match or not. If they decide to match, the firm and the worker bargain over the terms of an employment contract. The outcome of the bargain maximizes the Nash product of the gains from trade accruing to the worker and those accruing to the firm, where the exponent on the worker's gains from trade is γ and the exponent on the firm's gains from trade is $1 - \gamma$. The contingencies of the contract are rich enough to guarantee that the joint value of the match is maximized.

4.2 Definition of a BGP

The definition of a BGP is almost the same as in Section 2, except that A_t needs to be replaced by $\hat{A}_t = A_t N_t^{\beta}$ in all the equilibrium conditions, and the stationarity conditions for the unemployment rate u and for the distribution G_t of employed workers across matches of different qualities need to be amended to take into account the growth of the population.

Formally, the reservation quality R_t, the surplus of a match $S_t(z)$, and the tightness of the labor market θ satisfy the equilibrium conditions

$$R_t = \frac{b_t}{y_t} + \frac{\hat{A}_t}{y_t} p(\theta)\gamma \int_{R_t} S_t(\hat{z}) dF(\hat{z}), \tag{4.1}$$

$$rS_t(z) = y_t(z - R_t) + \mathring{S}_t(z), \tag{4.2}$$

$$k_t = \hat{A}_t q(\theta)(1-\gamma) \int_{R_t} S_t(\hat{z}) dF(\hat{z}). \tag{4.3}$$

The equilibrium conditions above are the same as (2.5), (2.9) and (2.10), except that $\hat{A}_t$ replaces A_t.

The UE rate, the EU rate and the unemployment rate u satisfy the stationarity conditions

$$\hat{A}_t p(\theta)[1 - F(R_t)] = h_{UE}, \tag{4.4}$$

$$G_t'(R_t)\mathring{R}_t = h_{EU}, \tag{4.5}$$

$$(1-u)h_{EU} + (1-u)g_N = uh_{UE}. \tag{4.6}$$

The stationarity conditions (4.4)-(4.5) for the UE and EU rates are the same as (2.11)-(2.12) with $\hat{A}_t$ replacing A_t. The stationarity condition (4.6) for the unemployment rate is different. The unemployment rate is stationary when the flow of workers from unemployment into employment, uh_{UE}, is equal to the flow of workers from employment into unemployment, $(1-u)h_{EU}$, plus the flow g_N of new workers into the labor market multiplied by the difference $1-u$ in the unemployment rate of new and old workers.

The condition guaranteeing that the distribution G_t grows at a constant rate g_z—in the sense that $G_t(z_t(x)) = G_0(z_0(x))$ where $z_t(x) = z_0(x)e^{g_z t}$ and $z_0(x)$ is the xth quantile of G_0—is

$$\begin{aligned} & (1-u)G_t'(z_t(x))z_t(x)g_z + u\hat{A}_t p(\theta)[F(z_t(x)) - F(R_t)] \\ = \; & (1-u)G_t'(R_t(x))R_t(x)g_z + (1-u)G_t(z_t(x))g_N. \end{aligned} \tag{4.7}$$

The inflow-outflow condition (4.7) is different than (2.15) because of the last term on the right-hand side. The left-hand side is the flow of workers into the group of those employed in a match of quality below the xth quantile, and it is given by the sum of the flow of employed workers whose match-quality z falls below the xth quantile and the flow of unemployed worker who enter a match with quality below the xth quantile. The first term on the right-hand side is the flow of workers employed in a match with quality below the xth quantile into unemployment. The second term is the flow g_N of workers entering the labor market times the difference between the fraction of the old workers who are employed at a match below the xth quantile, i.e. $(1-u)G_t(z_t(x))$, and the fraction of new workers who are employed at a match below the xth quantile, i.e. 0.

The above observations lead to the following definition of a BGP.

Definition 3: *A BGP is a tuple $\{R_t, S_t, \theta, h_{UE}, h_{EU}, u, G_t\}$ such that for all $t \geq 0$: (i) R_t, S_t and θ satisfy (4.1)-(4.3); (ii) h_{UE}, h_{EU} and u satisfy (4.4)-(4.6); (iii) G_t satisfies (4.7).*

4.3 Existence and properties of BGP

Following exactly the same steps as in Section 2, it is easy to prove that a BGP exists if and only if the distribution F of qualities of new matches is Pareto with coefficient α, the worker's flow income from being unemployed and the firm's flow cost from maintaining a vacancy both grow at the rate $g_y + (g_A + \beta g_N)/\alpha$, where $g_A + \beta g_N > 0$ is the growth rate of $\hat{A}_t$, and the discount factor r is greater than $g_y + (g_A + \beta g_N)/\alpha$. The restriction $g_A + \beta g_N > 0$ is required to guarantee that the reservation quality increases over time.

Formally, we can establish the following proposition.

Proposition 5 *(Existence and Uniqueness of BGP) Consider arbitrary growth rates g_y, g_A and g_N for the production technology, the search technology and population, with $g_y > 0$ and $g_A + \beta g_N > 0$. A BGP exists if and only if F is a Pareto distribution with coefficient $\alpha > 1$, the growth rates of unemployment income and vacancy costs are $g_b = g_k = g_y + (g_A + \beta g_N)/\alpha$, and the discount rate is $r > g_y + (g_A + \beta g_N)/\alpha$. If a BGP exists, it is unique and such that:*

(i) the unemployment rate u, the labor market tightness θ, the UE and EU rates are constant;

(ii) the initial distribution G_0 of employed workers is a Pareto truncated on the left at R_0;

(iii) the reservation quality R_t and the distribution G_t grow at the rate $(g_A + \beta g_N)/\alpha$;

(iv) labor productivity and aggregate output grow at the rate $g_y + (g_A + \beta g_N)/\alpha$.

In the BGP, the unemployment rate, the UE rate, the EU rate and the tightness of the labor market are all constant even though the size of labor market is continuously growing and the search process may display non-constant returns to scale. The necessary and sufficient conditions for the existence of a BGP when the size of the labor market grows and the returns to scale in the search process are non-constant are exactly the same conditions that are necessary and sufficient for the existence of a BGP when there are only improvements in the search technology. Namely, the distribution of qualities of new matches must be Pareto and vacancy costs and unemployment income must grow at the same rate as the economy.

The above finding is intuitive. The same mechanism that offsets the improvements in the search technology and keeps the unemployment, UE and EU rates constant over time also offsets the returns to scale in the search process, as both improvements in the search technology and returns to scale increase the rate at which workers and firms meet each other. The finding is important. It implies that if the conditions for the existence of a BGP hold—which we take to be the case given the stationarity of unemployment, UE, EU rates in the face of the enormous progress in information technology that has taken place during the last century—then one cannot measure the returns to scale to the search process by looking at how unemployment, UE and EU rates vary with the size of the workforce.

Is there any way, then, to measure the returns to scale in the search process? And is there some way to separately measure the contribution of improvements in the search technology and the contribution of returns to scale in the search process to the growth rate of the economy? The answer to these questions is positive. To see why this is the case, first note that the average productivity of labor is given by

$$\int_{R_t} y_t z G_t'(z)dz = e^{(g_y+g_A/\alpha+\beta g_N/\alpha)t}\int_{R_0} y_0 z G_0'(z)dz. \tag{4.8}$$

The above expression shows that the growth rate of the average productivity of labor is the sum of three terms. The first term is the growth rate g_y of the production technology. The second term is the growth rate g_A of the search technology divided by the tail coefficient α of the Pareto distribution F for the quality of new matches. This term captures the contribution of improvements in the search technology to the growth of the average productivity of labor. The third term is the growth rate g_N of the workforce multiplied by β/α, the ratio of the return to scale coefficient of the search process and the tail coefficient of the distribution F. This term captures the contribution of returns to scale to the growth of the average productivity of labor.

Next, note that the average number of workers considered by a firm before filling its vacancy is given by

$$\frac{\hat{A}_t q(\theta)}{\hat{A}_t q(\theta)[1-F(R_t)]} = \frac{e^{(g_A+\beta g_N)t}}{1-F(R_0)}. \tag{4.9}$$

The numerator on the left-hand side of (4.9) is the measure of workers considered by a firm for its vacancy in each unit of time. The denominator on the left-hand side of (4.9) is the rate at which a firm fills its vacancy. The number of workers considered by a firm for its vacancy in each unit of time divided by the rate at which a firm fills its vacancy gives us the average number of workers considered by a firm before filling its vacancy. The growth rate of the average number of workers considered for a vacancy is $g_A + \beta g_N$.

Third, consider two separate labor markets at the same point in time. The two markets are identical, except that the size of the workforce is different, $N_{1,t}$ and $N_{2,t}$. The vacancy costs, $k_{1,t}$ and $k_{2,t}$, and unemployment income, $b_{1,t}$ and $b_{2,t}$, are different but have the same constant of proportionality to the average labor productivity, i.e. $k_{1,t}/k_{2,t}$ and $b_{1,t}/b_{2,t}$ are both equal to $(N_{1,t}/N_{2,t})^{\beta/\alpha}$. These assumptions are consistent with our interpretation that opening a vacancy is a labor cost, and that the unemployment income is proportional to the average productivity and, hence, to the average wage in the market.

Under the conditions for a BGP, the two markets have the same unemployment rate, UE rate, EU rate, tightness and vacancies. Therefore, the returns to scale in the search process cannot be identified by examining the difference between these variables in markets 1 and 2. However, the returns to scale can be recovered by looking at the average number of applicants per each vacancy in market 1 relative to market 2, which is given by

$$\frac{A_t N_{1,t}^{\beta} q(\theta)}{A_t N_{1,t}^{\beta} q(\theta)[1-F(R_{1,t})]} \cdot \frac{A_t N_{2,t}^{\beta} q(\theta)[1-F(R_{2,t})]}{A_t N_{2,t}^{\beta} q(\theta)} = \frac{1-F(R_{2,t})}{1-F(R_{1,t})} = \left(\frac{N_{1,t}}{N_{2,t}}\right)^{\beta}, \tag{4.10}$$

where the last step makes use of the fact that $R_{1,t}/R_{2,t} = (N_{1,t}/N_{2,t})^{\beta/\alpha}$. The above expression implies that the difference in the log of applicants per vacancy is proportional to the difference in the log of the market size, where the constant of proportionality is the coefficient β on the return to scale in the search process.

Finally, the coefficient α of the Pareto distribution of qualities of new matches can be recovered from the cross-sectional distribution of wages. In particular, α is the tail coefficient of the cross-sectional distribution of wages for workers hired out of unemployment. Note that all of the measurements described above are valid not only for the model without search on the job described in this section, but also for a version of the model with on-the-job search.

We thus have the following identification strategy.

Proposition 6 *(Identification of returns to scale in search technology).*

(i) The coefficient β on the return to scale in the matching process can be measured as the coefficient in a regression of the log of applications-per-vacancies on the log of the size of the market.

(ii) The coefficient α of the Pareto distribution of qualities of new matches can be measured as the tail coefficient of the cross-section of wages for workers hired out of unemployment;

(iii) The contribution $g_N \beta/\alpha$ of returns to scale in search to the growth of the economy can be measured as the growth rate of population times β/α.

(iv) The contribution g_A/α of improvements in the search technology to the growth of the economy can be measured as the growth rate of applications-per-vacancy divided by α net of $g_N\beta/\alpha$.

5 Back-of-the-envelope calculations

We conclude the paper by trying to implement the identification strategy outlined in Proposition 6 to measure the contribution of improvements in the search technology and of returns to scale in the matching process to the growth rate of labor productivity. The identification strategy requires time-series and cross-sectional data on applications per vacancy, cross-sectional data on the wages of identical workers, and time-series for the average labor productivity and the size of the labor force.

As far as we know, there is no time-series for the average number of applicants per vacancy spanning more than a couple of years. However, Faberman and Menzio (2018) have a measure of applicants per vacancy in the US for the early 1980s, while Marinescu and Wolthoff (2016) and Faberman and Kudlyak (2016) have a measure of applicants per vacancy in the US for the early 2010s. The data used by Marinescu and Wolthoff (2016) also contains a detailed break-down of vacancies and their applications in different areas of the US. Measuring the wage distribution for identical workers is a task that we do not attempt, and instead we carry out our identification exercise for different values of the coefficient α of the Pareto distribution F. Given all the limitations of the analysis, it is fair to think of it as a back-of-the-envelope calculation. Yet, we believe that our findings clearly indicate that declining search frictions have has a sizeable contribution to the growth of productivity.

Faberman and Menzio (2018) analyze data from the Employment Occupation Pilot Project (EOPP), which is a survey of US firms that was carried out in 1980 and 1982 and contains information about characteristics of job openings (e.g., occupation, industry, location, etc...) and recruitment outcomes (e.g., number of applications per vacancy, number of interviews per vacancy, vacancy duration, wage paid to the hired worker, etc...). Faberman and Menzio (2018) find that the average number of applications per vacancy is 10.4 per week and that the average duration of a vacancy is 16.2 days. These figures suggest that the number of applications per vacancy is 24.

Marinescu and Wolthoff (2016) analyze proprietary data from CareerBuilder.com, which is the largest online job board in the US, and contains over 1 million jobs at each point in time and is visited by approximately 11 million unique job seekers during each month. For the sake of tractability, Marinescu and Wolthoff (2016) restrict attention to vacancies posted in the Chicago and Washington DC Designated Market Areas between January and March 2011. The data contains detailed information about job and firm characteristics (e.g., job title, occupation, name of the firm, industry, etc…) and outcomes (including applications per vacancy). They find that the average number of applications per vacancy is 59. Faberman and Kudlyak (2016) use proprietary data from SnagAJob.com, which is an online job search engine that focuses on hourly-paid job. They find that the average number of applications per vacancy is 31.

The data reported in Faberman and Menzio (2018), Marinescu and Wolthoff (2016) and Faberman and Kudlyak (2016) suggest that, from 1982 to 2011, the number of applications per vacancy has grown from 24 to somewhere between 31 and 59. Taking the average of 45, this is equivalent to an average growth rate of 2.2% per annum. Equation (4.9) implies that the growth rate of applications per vacancy is the sum of the growth rate in the search technology, i.e. g_A, and the return to scale in the search process caused by the increase in the size of the labor market, i.e. $g_N\beta$. Equation (4.8) implies that the contribution of the decline in search frictions, due to either technological progress or returns to scale, to the growth of the economy is given by the growth rate of applications per vacancies divided by the coefficient α of the Pareto distribution F. As mentioned above, measuring α is not easy, as it requires measuring the distribution of wages out of unemployment for workers that are inherently identical. This measurement exercise is outside the scope of our paper. However, α is unlikely to be very large. In fact, Postel-Vinay and Robin (2002) document—using a model that is similar to ours and allows for workers' heterogeneity that is unobserved by the econometrician—that search frictions account for almost half of the wage inequality among workers in the same occupation. For the sake of the argument, suppose that α was equal to 2. Then, declining search frictions would generate a growth in average labor productivity of 1.1 percentage points per annum. If α was equal to 4, declining search frictions would generate a growth in average labor productivity of 0.55 percentage point per annum. Even if α was equal to 8, declining search frictions would still generate a non-negligible growth in average labor productivity of 0.28 percentage points per year.

The proprietary data from CareerBuilder.com contains information of the number of applications per vacancy across the US and can be used to measure the returns to scale in the search process. Ioana Marinescu kindly agreed to run for us a regression of applications per vacancy on market size. In particular, she ran for us a regression of the log of applications per vacancy on the log of the population in the commuting zone of the vacancy. She finds that the regression coefficient on the log of the population size is 0.52. In light of (4.10), 0.52 is an estimate of the return to scale coefficient β in the search process. She finds a similar regression coefficient when she also controls for occupation. The regression implies that vacancies in a commuting zone that is 10% larger receive on average 5.2% more applications. Similarly, vacancies in a commuting zone of 10 million people would receive approximately 3 times more applications than vacancies in a commuting zone of 1 million people.

The estimate of β allows us to decompose the contribution of declining search frictions to the growth rate of the average productivity of labor. From 1982 to 2011, the US labor force went from 108 million to 152 million people, which is an average growth rate of 1.1% per annum. Since $\beta = 0.52$, this means that, of the annual growth rate of 2.2% in applications per vacancy, 0.6% percent is caused by increasing returns in the search process and the remaining 1.4% is caused by improvements in the search technology. In other words, approximately 1/4 of the growth in applications per vacancy is due to increasing returns to scale in search, and 3/4 is due to improvements in the search technology. In turn, this implies that 1/4 of the productivity growth generated by declining search frictions is due to increasing returns in the search process and 3/4 is due to improvements in the search technology.

The estimate of β has also interesting implications for understanding geographic differences in average labor productivity. For instance, the estimate of β implies that, in a commuting zone that is 10% larger, the average productivity of labor and, hence, wages would be 2.5% higher if $\alpha = 2$, 1.25% higher if $\alpha = 4$ and 0.62% higher if $\alpha = 8$ just because of increasing returns in the process by which firms and workers find each other. Similarly, in a commuting zone of 10 million people, relative to a commuting zone of 1 million people, average labor productivity and wages would be 50% higher if $\alpha = 2$, 25% higher if $\alpha = 4$ and 12.5% higher if $\alpha = 8$. Therefore, increasing returns to scale in the search process can explain a sizeable fraction of the productivity and wage differential across small and large cities. Yet, increasing returns to scale in the search process do not generate any differences in unemployment, UE and EU rates across small and large cities and, hence, cannot be detected by regressing the unemployment rate on the size of the city.

It is indeed well-documented that wages are systematically higher in larger cities but the unemployment rate is not systematically lower (see, e.g., Petrongolo and Pissarides 2006). The standard explanation for these two facts is that the search process features constant returns to scale—explaining why unemployment is independent of city size—and that, for some reason, workers are more productive in large cities than in small cities—explaining why wages are higher in larger cities. This standard explanation however is at odds with our findings on the number of applicants per vacancy across commuting areas of different size. Our explanation for these facts about geography is a simple corollary of the explanation for why unemployment, UE, EU rates, tightness and vacancies are constant over time in the face of enormous improvements in information technology. That is, if the distribution of qualities of new matches is Pareto, unemployment does not depend on improvements in the search technology and it does not depend on returns to scale in the search process because time-series and cross-sectional differences in the extent of search frictions are offset by differences in how selective firms and workers are with respect to which matches to form. Moreover, our explanation is consistent with (and, in fact, it is quantitatively disciplined by) the observation that applications per vacancy have increased over time and are systematically higher in larger commuting areas.

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