

# A Monetary Theory with Non-Degenerate Distributions\*

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## Abstract

We construct and analyze a tractable search model of money with a non-degenerate distribution of money holdings. We assume search to be directed in the sense that buyers know the terms of trade before visiting particular sellers. Directed search makes the monetary steady state block recursive in the sense that individuals' policy functions, value functions and the market tightness function are all independent of the distribution of individuals over money balances, although the distribution affects the aggregate activity by itself. Block recursivity enables us to characterize the equilibrium analytically. By adapting lattice-theoretic techniques, we characterize individuals' policy and value functions, and show that these functions satisfy the standard conditions of optimization. We prove that a unique monetary steady state exists. Moreover, we provide conditions under which the steady-state distribution of buyers over money balances is non-degenerate and analyze the properties of this distribution.

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## 1. Introduction

Monetary search theory originated in Kiyotaki and Wright (1989) provides a strong microfoundation of money by deriving positive value of fiat money from underlying frictions in decentralized exchange, such as the lack of double coincidence of wants and anonymity of individuals. The theory has undergone significant development and has been shown to be useful for addressing a wide range of monetary issues in theory and policy. Despite all the development, monetary search theory has had only limited success in exploring a natural outcome of decentralized exchange – a persistent, non-degenerate distribution of money balances. This distribution of money is important for both theory and policy. In this paper, we construct a tractable search model where a non-degenerate distribution of money can be persistent. We prove the existence of a unique monetary steady state, provide the conditions under which the distribution of money is non-degenerate, and examine the properties of the distribution.

Decentralized exchange naturally induces a non-degenerate distribution of money. Because matching is stochastic, different individuals may end up trading away different amounts of money. Even if all individuals hold the same amount of money initially, the distribution of buyers over money balances can fan out as decentralized exchange continues. In the first generation of search models (e.g., Kiyotaki and Wright, 1989) and the second generation (e.g., Shi, 1995, and Trejos and Wright, 1995), the distribution of money is made degenerate by the assumption that an individual can hold either zero or one unit of indivisible money. Search models in the third generation allow both money and goods to be divisible, but they impose other assumptions to make money distribution degenerate. Notably, Shi (1997) assumes that each household consists of a large number of members who share consumption and utility, and so all households hold the same amount of money in the equilibrium. Lagos and Wright (2005) assume that each decentralized market is followed by a centralized market in which individuals have quasi-linear preferences over a good, and so trading in the centralized market eliminates all dispersion in money holdings. Green and Zhou (1998, *GZ* henceforth) are the first to formally analyze a non-degenerate money distribution, but they assume that goods come as a fixed endowment and money can only be accumulated in discrete units. Without these restrictive assumptions, an analytically tractable microfoundation of money with a non-degenerate distribution has eluded monetary theory. Very little is known about even the fundamental properties of an equilibrium. For example, does such a steady state exist, is it unique, and when is the distribution non-degenerate?

A persistent, non-degenerate distribution of money is also important for policy analysis. One issue is the wealth effect of monetary policy. When individuals' money balances differ, so do

their marginal values of money. A change in monetary policy has a redistributive effect that can affect aggregate welfare. A related issue is the effect of monetary policy on real aggregate activities. As documented by Christiano et al. (1999) using vector autoregression (VAR), even an impulse monetary shock can have a persistent effect on aggregate real activity. This persistence seems difficult to be captured by models where the distribution of money is degenerate or not persistent, such as Lucas (1990). In contrast, Rotemberg (1984) follows Baumol (1952) and Tobin (1956) to model an individual's money holdings as the solution to a problem of optimal inventory management. Because cash withdrawals are staggered across individuals, the distribution of money is non-degenerate in the steady state, and a monetary shock affects aggregate real activity persistently. These papers, and the large literature spawned from them, are useful indications of the importance of money dispersion in analyzing policy, but they assume an exogenous role of money rather than deriving this role from the fundamentals of the model.<sup>1</sup>

We focus on the characterization of the steady state of a search economy with a non-degenerate distribution, although the broad research project is also motivated by the dynamic effect of monetary policy. This focus is a useful first step because it addresses the fundamental questions raised above on the equilibrium and provides a long-run anchor for a dynamic analysis. As a standalone contribution, the analysis provides a set of theoretical tools that help overcome the main difficulties in building a monetary theory with a non-degenerate distribution.

The main deviation from the literature of monetary search that makes our model tractable lies in the way we model search. The literature assumes search to be undirected in the sense that individuals do not know the terms of trade before they are matched. In contrast, we assume search to be directed in the sense that individuals know the terms of trade before a match, as in Peters (1991), Moen (1997), Acemoglu and Shimer (1999), Burdett et al. (2001) and Julien et al. (2000). Directed search is a realistic feature of an actual economy. It reflects the fact that individuals have information about the location and the price range of the goods they want to buy. They go directly to the sellers who sell the goods they want, rather than randomly search among all the sellers. In our model, each type of good can be sold in many submarkets that specify different terms of trade and tightness (i.e., the ratio of trading posts to buyers). Buyers choose which submarket to visit and firms choose how many trading posts to create in each

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<sup>1</sup>Shi (2003) extends the large-household framework in Shi (1997) to examine the persistence of a monetary shock with search frictions. He introduces long-term nominal bonds with a partial legal restriction that prohibits a fraction of the individuals such as government agents from accepting bonds as a means of payment for goods and services. A monetary shock has a more persistent real effect than in Lucas (1990) because the shock affects the composition of money and bonds used in the goods market in future periods. However, the persistence is not as strong as empirically documented, partly because the large-household framework makes the distribution of asset holdings across households degenerate.

submarket. There is a cost of creating a trading post for a period, and the number of trading posts in each submarket is determined by free entry. Once individuals are inside a submarket, a frictional matching process determines the matching probability for a trading post or a buyer as a function of the tightness of the submarket. In equilibrium, the tightness in each submarket is consistent with buyers' search choice and firms' creation of trading posts.

Directed search induces buyers to sort into submarkets. Specifically, because the marginal value of money is lower to a buyer who has a relatively high money balance, such a buyer has a strong desire to spend a relatively large amount of money on consumption and to spend it sooner rather than later. To do so, the buyer chooses to enter a submarket where he has a relatively high matching probability to trade a relatively large amount of money for a large quantity of goods. Firms cater to this desire by creating a relatively large number of trading posts per buyer in this submarket. Because buyers with different money holdings choose not to mix with each other, a buyer's optimal choices depend on the buyer's own money balance and the tightness of the particular submarket he visits, but not on the distribution of individuals over money balances. Moreover, because each submarket is tailored to only one group of buyers with a particular money balance, the tightness of each submarket that ensures zero profit for a trading post does not depend on the distribution of money. Precisely, individuals' policy functions, value functions and the market tightness function are all independent of the distribution in the steady state. This feature of the steady state is referred to as *block recursivity*.

With block recursivity, the distribution of money ceases to be part of the state space in individuals' decision problems. This overcomes the main roadblock of tractability that arises in the literature when the endogenous distribution of money affects individuals' decisions. As a result, we can characterize an individual's policy and value functions solely as functions of the individual's own balance. Having done so for each balance separately, we can compute the flows of individuals across money balances to obtain the distribution. The analytical tractability of the model enables us to prove that a unique monetary steady state exists, to determine when the steady-state distribution of buyers over money balances is non-degenerate, and to analyze the properties of this distribution.

In the steady state, the support of the distribution consists of a finite number of values of money balance, each of which is associated with one active submarket. Moreover, an individual goes through purchasing cycles. When the individual has no money, he works to obtain money and then becomes a buyer. Starting with a high balance, a buyer enters a submarket where he has a high matching probability, spends a large amount of money and obtains a large quantity of goods. For the next trade, the buyer will go into a submarket where his matching probability,

the required spending and the quantity of goods obtained in a trade are all lower. The buyer will continue this pattern until he depletes his balance, at which point he will work again.

The unique monetary steady state unifies the literature by nesting two well-known classes of models as special cases. In one case, the distribution of money is degenerate in the steady state, which occurs when individuals are sufficiently impatient. In this case, all buyers hold the same amount of money and spend the entire amount in one trade, and so a purchasing cycle consists of only one purchase. This endogenous pattern resembles the one assumed in the models with indivisible money (e.g., Shi, 1995, and Trejos and Wright, 1995). However, the endogenously generated pattern has a very different policy effect from the exogenously assumed pattern. Namely, a one-time change in the money stock does not affect the real activity in the steady state in our model, but it does in Shi (1995), Trejos and Wright (1995) and GZ.

The other case of the unique steady state features a non-degenerate distribution of buyers over money balances. This case arises when individuals are sufficiently patient, together with other conditions specified in Theorem 4.2. In this case, each buyer runs down money balance in a purchasing cycle through consecutive purchases, and the purchasing pattern is staggered across the buyers. Moreover, because the buyers who hold a high balance trade relatively quickly and exit from that balance, there are more buyers with low balances than with high balances, and so the density of the distribution in a purchasing cycle is a decreasing function of money balance. The purchasing cycle resembles the one in the inventory model of money by Baumol (1952) and Tobin (1956), but the latter authors model the role of money in a reduced form. The decreasing density of money distribution resembles that in GZ, but GZ restrict that goods come as a fixed endowment and money can only be accumulated in discrete units. Moreover, in contrast to both Baumol-Tobin and GZ, we allow individuals to choose among submarkets that differ in the terms of trade and the matching probability. As a realistic feature, this endogeneity should be important for how monetary policy affects the aggregate activity. We will contrast our model further to the Baumol-Tobin model in subsection 4.1 and to GZ in subsection 4.3.

A large part of this paper is devoted to the analysis of a buyer's decision problem, which establishes the properties of the policy and value functions. We provide a set of analytical tools to overcome some difficulties in the use of dynamic programming. The difficulties arise from the features that a buyer's objective function is not concave and that a buyer's value function cannot be assumed to be differentiable a priori. These difficulties prevent us from using the standard approach in dynamic programming (e.g., Stokey et al., 1989) to analyze the policy and value functions. In subsection 3.2.1, we will give an overview of these difficulties and the way in which we resolve them. A short description is that we adapt lattice-theoretic techniques (see

Topkis, 1998) to prove that a buyer’s policy functions are monotone functions of the real balance. Using this result, we prove further that optimal choices obey the first-order conditions, the value functions are differentiable and the envelope conditions hold. By establishing these standard conditions formally, we hope to make the model easy to use. This procedure of analyzing a dynamic programming problem is of independent interest because it is applicable in a variety of dynamic models that involve both discrete and continuous choices.<sup>2</sup>

It is important to clarify that our analysis and the main results do not follow simply from the labor search literature, despite the fact that this literature has explored directed search and block recursivity (e.g., Shi, 2009, Gonzalez and Shi, 2010, and Menzio and Shi, 2011). Several elements in our model are important for monetary theory, but not necessarily so for labor theory. First, an individual’s gain from a monetary trade depends not only on how the match surplus is split, but also on how all individuals in the economy value money. A monetary equilibrium must determine this value of money. This is not necessary in a labor search model because the value of a match is determined by preferences there. Second, money balance is a stock variable that can be accumulated or decumulated over time through trade, and there is a market clearing condition on the aggregate stock of money. Such a stock variable has been absent in many labor-search models. Third, a buyer in a monetary model optimally chooses the length of a purchasing cycle and the amount of money to be spent in each period within a cycle. Most parts of this paper (e.g., sections 3 and 4) are devoted to resolving these monetary issues. Even at the technical level of using lattice-theoretic techniques, our analysis differs from that in Gonzalez and Shi (2010). While Gonzalez and Shi explore convexity of the value function to apply lattice-theoretic techniques, a buyer’s value function in our model is neither convex nor concave. To apply lattice-theoretic techniques, we analyze a buyer’s decision problem in steps (see subsection 3.2.2).

In the literature of monetary search, Corbae et al. (2003) seem the first to incorporate directed search. They focus on the formation of trading coalitions and assume that money and goods are indivisible. Rocheteau and Wright (2005) examine directed search as a robustness check, Lagos and Rocheteau (2005) compare directed search with bargaining in the welfare effect of inflation, and Galenianos and Kircher (2008) and Julien et al. (2008) examine directed search with auctions. These papers do not formulate a block recursive equilibrium. Moreover, they follow Lagos and Wright (2005) to assume quasi-linear preferences that make either money distribution degenerate or its real effect temporary.<sup>3</sup> In addition, Lagos and Rocheteau (2005) apply lattice-theoretic

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<sup>2</sup>There are a large number of applications of lattice-theoretic techniques in dynamic programming and, specifically, in models of economic growth. Gonzalez and Shi (2010) use these techniques in a labor search model and provide a partial list of references.

<sup>3</sup>Berentsen, Camera and Waller (2005) extend the Lagos-Wright (2005) model to allow for two rounds of trading

techniques to examine comparative statics. However, their assumption of quasi-linear preferences implies that the future value function is linear in money balance. This feature sidesteps the analytical difficulties overcome here that are intrinsic to search models where money has a wealth effect (see subsection 3.2.1).

With undirected search, some papers have studied a non-degenerate money distribution. GZ are the first along this line, but they restrict that goods come as a fixed endowment and money can only be accumulated in discrete units. Zhou (1999) extends the GZ model by introducing a production cost. Berentsen et al. (2004) introduce lotteries into GZ. Zhu (2005) extends the GZ model by making goods divisible. He studies a sequence of economies with discrete money holdings and characterizes the limit where the size of the discreteness goes to zero. Eliminating the indivisibility restrictions, Molico (2006) and Chiu and Molico (2008) numerically compute the equilibrium. In particular, Chiu and Molico (2008) generate a non-degenerate money distribution by extending the Lagos-Wright model to allow the cost function in the centralized market to be strictly convex. These numerical exercises are useful, but they do not address the fundamental issues about an equilibrium with a non-degenerate money distribution that have eluded the literature, such as existence and uniqueness of the steady state. These models are analytically intractable precisely because the assumption of undirected search makes the equilibrium not block recursive. Even for a quantitative analysis, our model is easier to compute than these models of undirected search, as we will discuss in section 6. Moreover, undirected search models, including Molico (2006) and Chiu and Molico (2008), have several implications in contrast with our model, which we will discuss at the end of subsection 2.1.

## 2. A Monetary Economy with Directed Search

### 2.1. The model environment

There are  $I$  types of individuals and  $I$  types of perishable goods indexed by  $i \in \{1, 2, \dots, I\}$ , where  $I \geq 3$ . Each type  $i$  consists of a continuum of individuals with measure one who are specialized in the consumption of good  $i$  and the production of good  $i + 1$  (modulo  $I$ ). The preferences of a type  $i$  individual are represented by the utility function  $\sum_{t=0}^{\infty} \beta^t [U(q_t) - h(\ell_t)]$ , where  $\beta \in (0, 1)$  is the discount factor,  $U : \mathbb{R}_+ \rightarrow \mathbb{R}$  is the utility of consumption of good  $i$ , and  $h : [0, 1] \rightarrow \mathbb{R}$  is the disutility of labor. We assume that  $U$  is strictly increasing, strictly concave and twice

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in the decentralized market before a centralized market opens for a homogeneous good over which individuals have quasi-linear preferences. The non-degenerate distribution of money is not persistent, because it is re-set frequently by the trading in the centralized market. An extension of the Lagos-Wright model to allow for many consecutive rounds of decentralized trading is not tractable. Faig (2008) introduces lotteries into the Lagos-Wright model to partially avoid the use of quasi-linear preferences. Again, the money distribution is degenerate in his model.

continuously differentiable, with the boundary properties:  $U(0) = 0$ ,  $U'(\infty) = 0$ , and  $U'(0)$  is sufficiently large. Similarly, we assume that  $h$  is strictly increasing, strictly convex and twice continuously differentiable, with the boundary properties:  $h(0) = 0$  and  $h'(1) = \infty$ . In addition to consumption goods, there is an object called fiat money which is intrinsically worthless, perfectly divisible and costlessly storable. The supply of fiat money per capita is  $M$ . We assume that  $M$  is constant in most parts of this paper and will allow for money growth in section 5.

The economy is also populated by  $I$  types of firms. Each type  $i$  consists of a large number of firms that are specialized in the production and distribution of good  $i$ . A type  $i$  firm operates a technology of constant returns to scale that transforms any amount of labor supplied by individuals of type  $i - 1$  (modulo  $I$ ) into that amount of good  $i$ .<sup>4</sup> Moreover, a type  $i$  firm can open a trading post in the market for good  $i$  using  $k > 0$  units of labor supplied by individuals of type  $i - 1$  (modulo  $I$ ). Firms are owned by the individuals through a balanced mutual fund.

In every period, a labor market and a product market open. Firms can participate in both markets in the same period. In contrast, individuals can participate in either the labor market or the product market. That is, in a given period, individuals must choose whether to become workers or buyers. Before making this choice, individuals can play a fair lottery. Even though individuals are risk averse, a lottery can be desirable because the value function without the lottery can be non-concave at particular money balances. One cause of non-concavity is the discrete nature of the decision on which market to enter. Another cause is the tradeoff between the matching probability and the surplus of trade in the product market, to be described later.

The labor market is centralized and frictionless. Taking the nominal wage rate as given, each firm chooses how much labor to demand and each worker chooses how much labor to supply. In equilibrium, the nominal wage rate equates the demand for and the supply of labor of each type. All labor is paid at the end of the period from the proceeds of sales. Workers are paid in money instead of goods because they do not want to consume the good produced by the firm in which they work and because goods are perishable between periods. Moreover, an IOU issued by a firm that promises future repayment is not accepted, because the firm is better off exiting the market than honoring such IOUs.<sup>5</sup>

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<sup>4</sup>The linear production technology is assumed without loss of generality. Note that the disutility of labor supply,  $h(\ell)$ , is strictly convex and the utility of consumption is strictly concave.

<sup>5</sup>Firms are assumed to commit to the wage payment at the end of the period. Similar assumptions of within-period commitments are required in all models of decentralized exchange where goods are perishable, such as Shi (1995) and Trejos and Wright (1995). In the latter models, the two traders in a match are assumed to commit to the delivery of money and goods according to the bargained terms of trade. Without this assumption, a buyer would renege and ask the seller to give up the goods for free once the goods are produced. Although a detailed description of the environment can be given to justify the within-period commitment, it is often skipped in the literature, as it is here, because it would take the attention away from the main issues.



To simplify the notation, we choose labor, instead of goods or money, as the numeraire in this model. Let  $\omega M$  be the nominal wage rate, and so one unit of money is worth  $1/(\omega M)$  units of labor. We refer to the quantity of money expressed in terms of labor as the *real balance*. Thus, the real balance per capita in the economy is equal to  $1/\omega$ . We will also express the price of goods in labor units later. Although  $\omega$  is normalized by the money stock, we shorten the phrase by referring to  $\omega$  as the nominal wage rate whenever there is no confusion.

The product market is decentralized and has search frictions. Buyers and trading posts meet in pairs and there is no record keeping of their actions once they exit a trade. The market for each type  $i$  goods is organized in a continuum of submarkets indexed by the terms of trade  $(x, q) \in \mathbb{R}_+ \times \mathbb{R}_+$ , where  $x$  is the real balance paid by the buyer and  $q$  is the quantity of goods obtained by the buyer in a trade. A firm chooses how many trading posts to create in each submarket, and a buyer chooses which submarket to visit. As is standard in search models, the length of a period is such that a buyer can visit at most one submarket in a period. The matching process is frictional in each submarket. Let  $\theta$  denote the *tightness*, i.e., the ratio of trading posts to buyers, in a submarket. In a submarket with tightness  $\theta$ , a buyer is matched with probability  $b = \lambda(\theta)$ , and a trading post is matched with probability  $s = \rho(\theta)$ . The function  $\lambda : \mathbb{R}_+ \rightarrow [0, 1]$  is a strictly increasing function with boundary conditions  $\lambda(0) = 0$  and  $\lambda(\infty) = 1$ . The function  $\rho : \mathbb{R}_+ \rightarrow [0, 1]$  is a strictly decreasing function such that  $\rho(\theta) = \lambda(\theta)/\theta$ ,  $\rho(0) = 1$  and  $\rho(\infty) = 0$ . Since  $b$  and  $s$  are both functions of  $\theta$ , we can express a trading post's matching probability as a function of a buyer's matching probability:  $s = \mu(b) \equiv \rho(\lambda^{-1}(b))$ . Clearly,  $\mu(b)$  is a decreasing function. We assume that  $1/\mu(b)$  is strictly convex in  $b$ .

Because firms and buyers choose which submarket to enter, a type  $i$  buyer will choose to participate only in the submarkets where type  $i$  goods are produced, i.e., where trading posts are created by type  $i$  firms. Moreover, across the submarkets that sell the same good, search is directed as in Moen (1997), Acemoglu and Shimer (1999), Burdett et al. (2001) and Shi (2001). That is, firms and buyers take into account the fact that market tightness varies with the terms of trade across the submarkets according to a function  $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . The function  $\theta$  is endogenously determined in the equilibrium by the requirement that  $\theta(x, q)$  be equal to the ratio of trading posts to buyers in submarket  $(x, q)$  for all  $(x, q)$ . As a result, matching probabilities,  $b$  and  $s$ , are endogenous functions of  $(x, q)$ .<sup>6</sup>

When a buyer meets a trading post in submarket  $(x, q)$ , the buyer pays the real balance  $x$  for  $q$

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<sup>6</sup>Note that the price of goods in a submarket alone is not adequate for describing a submarket because a buyer may not spend all the money in a trade. Also, in subsection 2.4, we will briefly contrast directed search with a perfectly competitive goods market and a trading-post model with no search frictions.

units of the consumption good. The buyer must pay the seller with money because neither barter nor credit is feasible. The buyer cannot pay the seller with goods because goods are perishable and there is no double coincidence of wants in goods between the buyer and the seller. Moreover, the buyer cannot pay the seller with an IOU because individuals are anonymous; once they exit a trade, they can renege on their IOUs without fear of retribution. Thus, the amount of money that a buyer can spend in a trade is bounded above by the balance he carries into the trade.

The environment above retains many standard features of a search model. In particular, matching is stochastic in each submarket, an individual can perform only one role in a period, and a buyer has at most one match in a period. Directed search is the main difference of our environment from the literature of monetary search. In order to direct search, some commitment is needed for individuals to make a meaningful tradeoff between the surplus of a trade and the matching probability. In our paper, the commitment is to the posted terms of trade. Let us clarify this assumption with three remarks. First, directed search is a realistic feature of many markets in an actual economy. The trading posts in a submarket may be interpreted as a collection of stores in a particular location or sellers who offer similar terms of trade. Buyers often choose the location and the price range of goods at which they shop, rather than randomly search among all the sellers. Second, modeling a market as a collection of trading posts can be traced at least back to Shapley and Shubik (1977), who argue that the setup is useful for retaining the force of competition while allowing for coordination frictions.<sup>7</sup> Third, commitment to the terms of trade is a simple way to model directed search, but it is not the only one. Search can be directed by commitment to a variety of trading arrangements, from posted prices (e.g., Burdett et al., 2001) to auctions (Julien et al., 2000).

Directed search has other desirable implications for modeling monetary exchange, which contrast with undirected search models (e.g., Shi, 1997, Lagos and Wright, 2005, Molico, 2006, and Chiu and Molico, 2008). First, undirected search models have often (although not always) assumed that the individuals in a match bargain over the terms of trade. To characterize the bargaining outcome simply, the models require a strong assumption that individuals' money holdings are public information. If money holdings are private information, instead, the bargaining game can have multiple equilibria, which complicate the analysis significantly and weaken the predictive power. Our model does not suffer from this problem. Because search is directed, whether an individual's money balance is public or private information does not matter to the analysis. In

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<sup>7</sup>The main difference between directed search and the Shapley-Shubik setup is that with directed search, matching is frictional and stochastic inside each submarket, and so there is a trade-off between the terms of trade and the matching probability. In the Shapley-Shubik setup, all participants at a post succeed in trade, provided that the number of participants on each side of the post is strictly positive.

any submarket, the expected payoff to an individual is determined by the terms of trade and the individual's own money balance. An individual can calculate this expected payoff without the need to know how much money the trading partner holds. Second, with undirected search and bargaining (under public information), a buyer with a higher balance pays a higher price in the equilibrium not because the buyer optimally chooses so, *ex ante*, but rather because the buyer is held up by the seller, *ex post*. In our model, as will be proven later, a buyer with a higher balance optimally chooses to pay a higher price because the submarket with a higher price compensates the buyer with a higher matching probability.

## 2.2. An individual's decision

Let  $V(m)$  denote the lifetime utility of an individual who starts a period with the real balance  $m$  (expressed in units of labor). We refer to  $V$  as the *ex-ante value function*, since it is measured before the individual makes any decision in a period. Let  $B(m)$  denote a *buyer's value function*, i.e., the lifetime utility of an individual who enters the product market as a buyer with the real balance  $m$ . Similarly, let  $W(m)$  denote a *worker's value function*, i.e., the lifetime utility of an individual who enters the labor market as a worker with the real balance  $m$ .

A worker chooses labor supply,  $\ell$ , which generates  $\ell$  units of the real balance as wage income. The individual also owns a diversified portfolio of the firms. However, the return to this portfolio is zero since all firms earn zero profit in the equilibrium. Thus, a worker who enters the labor market with a real balance  $m$  will have a real balance  $m + \ell$  at the end of the period. The discounted value of this balance is  $\beta V(m + \ell)$ . The worker's value function,  $W$ , obeys:

$$W(m) = \max_{\ell \in [0,1]} [\beta V(m + \ell) - h(\ell)]. \quad (2.1)$$

Denote the optimal choice of  $\ell$  as  $\ell^*(m)$  and the implied real balance at the end of the period as  $y^*(m) = m + \ell^*(m)$ . We refer to  $\ell^*(\cdot)$  and  $y^*(\cdot)$  as a worker's policy functions.

A buyer chooses which submarket  $(x, q)$  to enter, taking the tightness function  $\theta(x, q)$  as given. In submarket  $(x, q)$ , the buyer will meet a trading post with probability  $\lambda(\theta(x, q))$ , in which case he will trade a real balance  $x$  for  $q$  units of goods. Current consumption yields utility  $U(q)$ , and the residual balance  $(m - x)$  yields the discounted value  $\beta V(m - x)$ . With probability  $1 - \lambda(\theta(x, q))$ , the buyer will not have a match, in which case he will retain the balance  $m$  whose discounted value will be  $\beta V(m)$ . Thus, the buyer's value function,  $B$ , obeys:

$$B(m) = \max_{x \in [0, m], q \geq 0} \{ \lambda(\theta(x, q)) [U(q) + \beta V(m - x)] + [1 - \lambda(\theta(x, q))] \beta V(m) \}. \quad (2.2)$$

The buyer's optimal choices are represented by the policy functions  $(x^*(m), q^*(m))$ .

An individual chooses whether to be a worker or a buyer in the period. This choice induces:

$$\tilde{V}(m) = \max\{W(m), B(m)\}. \quad (2.3)$$

Notice that  $\tilde{V}$  may be non-concave for some real balances, even when  $W$  and  $B$  are concave functions. Thus, there is a potential gain to the individual from playing fair lotteries before making the above choice on whether to be a worker or a buyer. Denote a lottery as  $(z_j, \pi_j)_{j=1,2}$ , with  $z_2 \geq z_1$ , where  $\pi_j$  is the probability that the prize  $z_j$  is realized. The lifetime utility of the prize  $z_j$  is  $\tilde{V}(z_j)$ . Thus, the ex ante value function induced by the lottery choice is:

$$V(m) = \max_{(z_1, z_2, \pi_1, \pi_2)} \left[ \pi_1 \tilde{V}(z_1) + \pi_2 \tilde{V}(z_2) \right] \quad (2.4)$$

$$\text{s.t. } \begin{aligned} \pi_1 z_1 + \pi_2 z_2 &= m, & \pi_1 + \pi_2 &= 1, & z_2 &\geq z_1, \\ \pi_j &\in [0, 1] & \text{and } z_j &\geq 0 & \text{for } j &= 1, 2. \end{aligned}$$

Let  $(z_j^*(m), \pi_j^*(m))_{j=1,2}$  denote the individual's optimal choice of a lottery.<sup>8</sup>

### 2.3. A firm's decision

A firm chooses how many trading posts to create in each submarket and how much labor to employ, taking as given the wage, 1, and the tightness function,  $\theta(x, q)$ . The firm hires labor to create trading posts and produce goods, and pays labor with money received from selling goods. Consider submarket  $(x, q)$ . The cost of creating a trading post is  $k$  units of labor. A trading post in submarket  $(x, q)$  will have a match with probability  $\rho(\theta(x, q))$ , in which case the firm uses  $q$  units of labor to produce  $q$  units of goods and exchanges for a real balance  $x$ . Thus, the expected benefit of creating a trading post in submarket  $(x, q)$  is  $\rho(\theta(x, q))(x - q)$ . If  $\rho(\theta(x, q))(x - q) < k$ , it is optimal for the firm not to create any trading post in submarket  $(x, q)$ . If  $\rho(\theta(x, q))(x - q) > k$ , it is optimal for the firm to create infinitely many trading posts in submarket  $(x, q)$ . If  $\rho(\theta(x, q))(x - q) = k$ , the firm is indifferent between creating different numbers of trading posts in submarket  $(x, q)$ . The case  $\rho(\theta(x, q))(x - q) > k$  never occurs, because it implies  $\theta(x, q) = \infty$  and, hence,  $\rho(\theta(x, q)) = 0$ , which contradicts the condition for the case. Thus, in any submarket  $(x, q)$  visited by a positive number of buyers, the tightness is consistent with the firm's incentive to create trading posts if and only if

$$\rho(\theta(x, q))(x - q) \leq k \quad \text{and} \quad \theta(x, q) \geq 0, \quad (2.5)$$

---

<sup>8</sup>Because a lottery is defined for any given  $m$ , it is used by individuals who hold the *same* balance. Thus, a lottery is not introduced here for individuals with heterogeneous holdings to exchange their balances among each other between trading rounds. Moreover, a lottery may or may not be played in the equilibrium and, when it is played in the equilibrium, it is played at only one level of money balance (see Lemma 4.1).

where the two inequalities hold with complementary slackness. In any submarket  $(x, q)$  that is not visited by buyers, the tightness can be arbitrary if  $k$  is greater than  $\rho(\theta(x, q))(x - q)$ . However, following Shi (2009), Menzio and Shi (2010, 2011) and Gonzalez and Shi (2010), we restrict attention to equilibria in which (2.5) also holds for such submarkets.<sup>9</sup>

Note that (2.5) implies that aggregate profit is zero. If  $N(x, q)$  is the distribution of trading posts across submarkets, then the sum of money received from sales,  $\int \rho(\theta(x, q))x dN(x, q)$ , exactly covers the sum of money paid to the workers,  $\int [k + \rho(\theta(x, q))q] dN(x, q)$ . This implies that expected profit of each firm is zero. Basing on this result, we assume that actual profit of each firm is also zero. To justify this assumption, one may imagine that the number of firms is finite and each firm creates a large number of trading posts so that the law of large numbers applies to each firm to guarantee deterministic revenue and cost for each firm. Although each firm in this case has some size in the market, it is not uncommon to assume firms as price takers in the labor market as we do. For example, the celebrated work of Debreu (1959) assumes that a finite number of firms and a finite number of households take prices parametrically, even though their actions affect equilibrium prices.

## 2.4. Equilibrium definition and block recursivity

We define a monetary steady state as follows:

**Definition 2.1.** *A monetary steady state consists of value functions,  $(V, W, B)$ , policy functions,  $(\ell^*, x^*, q^*, z^*, \pi^*)$ , market tightness function  $\theta$ , a wage rate  $\omega$ , and a distribution of individuals over real balances,  $G$ , that satisfy the following requirements:*

- (i)  *$W$  satisfies (2.1) with  $\ell^*$  as the associated policy function;*
- (ii)  *$B$  satisfies (2.2) with  $(x^*, q^*)$  as the associated policy functions;*
- (iii)  *$V$  satisfies (2.4) with  $(z^*, \pi^*)$  as the associated policy functions;*
- (iv)  *$\theta$  satisfies (2.5) for all  $(x, q) \in \mathbb{R}_+^2$ ;*
- (v)  *$G$  is the ergodic distribution generated by  $(\ell^*, x^*, q^*, z^*, \pi^*, \theta)$ ,*<sup>10</sup>
- (vi)  *$\omega$  is such that  $\omega < \infty$  and  $\int m dG(m) = 1/\omega$ .*

---

<sup>9</sup>This restriction on the beliefs out of the equilibrium “completes” the market in the following sense: A submarket is inactive only if, given that some buyers are present in the submarket, the expected benefit to a lone trading post in the submarket is still lower than the cost of the trading post. This restriction can be justified by a “trembling-hand” argument that a small measure of buyers appear in every submarket exogenously. Similar restrictions are common in the literature on directed search, e.g., Moen (1997) and Acemoglu and Shimer (1999).

<sup>10</sup>The general specification of the law of motion of  $G$  is cumbersome at this point and not necessary for the equilibrium analysis. In section 4 we will characterize the law of motion of  $G$  implied by optimal choices.

Requirements (i)-(iv) are explained in previous subsections. Requirement (v) asks the distribution of individuals over real balances to be stationary and consistent with the flows of individuals induced by optimal choices. Requirement (vi) asks that money should have a positive value and that all money should be held by the individuals. Specifically, the sum of real balances according to the distribution  $G$  must be equal to the total real balance in the economy,  $1/\omega$ . We did not specify the labor market clearing condition in the above definition, because such a condition is implied by requirement (vi) in a closed economy.

As we will show in section 4, the support of the equilibrium distribution is a discrete set. However, for individuals to optimally choose to hold only the balances in this set, they need to know the optimal choice and the payoff of holding any balance outside the equilibrium set. This information is provided by the policy and value functions in (i)-(iii) above. Similarly, although the number of submarkets participated by a positive measure of individuals is finite in the equilibrium, individuals need to know the tightness in all submarkets, given by the function  $\theta$  in (iv) above, in order to choose optimally which submarket to participate in (see footnote 9).

Equilibrium objects and requirements in Definition 2.1 can be grouped into two blocks. The first block consists of the value functions, the policy functions and the market tightness function, which are determined by requirements (i) - (iv). The second block consists of the distribution of individuals over real balances and the wage rate, which are determined by requirements (v) and (vi). The second block depends on the objects in the first block, but the first block is not affected by the second block. That is, the value functions, the policy functions and the market tightness function are independent of the distribution and the wage rate. We refer to this property of the equilibrium as *block recursivity*, a phrase coined by Shi (2009), Menzio and Shi (2010, 2011), and Gonzalez and Shi (2010). Clearly, even when an equilibrium is block recursive, the distribution is important because it affects the aggregate activity.

Block recursivity is an attractive property of our model because it enables us to solve for equilibrium value functions, policy functions and the market tightness function without solving for the distribution of individuals over real balances. After obtaining the objects in the first block, we can compute the equilibrium distribution by simply equating the flows of individuals into and out of each level of the real balance. Thus, the steady state is tractable even when the distribution of real balances is non-degenerate. In contrast, when the distribution is an aggregate state variable that appears in the policy and value functions, one must compute the objects in the two blocks simultaneously and, since the distribution is endogenous and potentially has a large dimension, the computation of an equilibrium is complicated. In fact, it is to circumvent this complexity that monetary models have imposed assumptions on the model environment to

make the distribution degenerate (e.g., Shi, 1997, Lagos and Wright, 2005).

Directed search and free-entry of trading posts are responsible for the steady state to be block recursive. With directed search, individuals choose to enter only the submarkets with the best tradeoff between the terms of trade and the matching probability, as formulated in subsection 2.2. The tightness function provides all information on the market that is relevant for this tradeoff. Given the tightness in each submarket, a buyer's optimal decision on which submarket to visit depends only on the buyer's own real balance, and not on how real balances are distributed among other buyers. Similarly, given the tightness in each submarket, the expected profit of a trading post in a submarket depends only on the particular real balance of the buyers who are expected to enter that submarket, and not on how real balances are distributed in other submarkets. In turn, the tightness in each submarket is determined by free-entry of trading posts, which drives the expected profit of a trading post down to zero wherever  $x - q \geq k$ , and the tightness to zero wherever  $x - q < k$ . Because the expected profit of a trading post in each submarket depends only on the real balance of the buyers who will enter that submarket, so does the resulting tightness. Thus, value functions, policy functions, and the market tightness function are all independent of money distribution in the steady state.

To appreciate the role of directed search, consider an environment with undirected search in which all buyers and trading posts go through the same random matching process first and then decide whether to trade. The terms of trade can be either posted before the meeting (without serving the function of directing search) or bargained after the meeting. If the terms of trade are posted before a meeting takes place, whether a particular match generates a non-negative surplus depends on the real balance of the buyer in the match. Because the buyer is randomly drawn, the trading probability depends on the distribution of buyers over real balances. If the terms of trade are instead bargained after a meeting takes place, they depend on real balances of both individuals in the match which are randomly drawn from the distribution. In both cases, the distribution of individuals over real balances affects the value function and the expected benefit of a trading post. Because the tightness of the market is such that the expected benefit of a trading post is equal to the cost, the tightness is also a function of the distribution. That is, when search is undirected, the equilibrium is not block recursive.

Finally, let us remark on the assumption that the labor market is perfectly competitive. Although this assumption is standard in macro, it is not used in most money-search models which, instead, assume that a worker's income depends on the outcome of random matching (e.g., Shi, 1995, and Trejos and Wright, 1995). In our model, each firm hires workers to produce goods and to maintain a large number of trading posts. Although each trading post may or

may not have a trade, the law of large numbers implies that a firm's total revenue from all trading posts together is deterministic, which enables the firm to pay a deterministic competitive wage rate. Moreover, we will show in section 4 that all workers go to work with zero balance. However, neither the deterministic wage income nor the degenerate real balance among workers in the equilibrium is important for block recursivity. This should be clear from the fact that block recursivity is a statement about the independence of the entire value functions, policy functions and the tightness function on the distribution, not just the independence at particular real balances. For example, if there are idiosyncratic shocks to the disutility of labor or a worker's matching outcome, then the distribution of wage income among workers will be non-degenerate in the equilibrium. But the equilibrium will still be defined in the same way as above and will remain block recursive. We abstract from such heterogeneity among workers in order to focus on a buyer's decision and money distribution among the buyers.

### 3. Equilibrium Policy and Value Functions

In this section we establish existence, uniqueness and other features of value and policy functions. A center piece of this analysis is subsection 3.2 on a buyer's value and policy functions. We prove that a buyer's policy functions are monotone, which implies that buyers sort into different submarkets according to the real balance. A buyer with a higher balance chooses to search in a submarket where he can spend a larger balance and get a higher quantity of goods. Such a buyer also has a higher matching probability. Sorting leads to a stylized pattern of purchases and a clear characterization of the equilibrium in section 4.

Monotonicity of policy functions is also critical for us to prove that the standard conditions of optimization, such as the first-order conditions and the envelope conditions, hold in our model. The characterization of a buyer's problem is technically challenging because the problem is not well-behaved. In fact, a buyer's objective function is not concave in the choice and state variables jointly. We cannot use standard arguments in dynamic programming (e.g., Stokey et al., 1989) to establish monotonicity of the policy functions and differentiability of the value function and, in turn, to establish the validity of the envelope and first-order conditions. Instead, we develop an alternative set of arguments that first prove monotonicity of the policy functions, then differentiability of the value function and finally the validity of the first-order and envelope conditions. These arguments are of independent interest because they are likely to apply to a variety of dynamic models that involve both discrete and continuous choices.

A map of the analysis in this section is as follows. First, we assume that an individual's real



balance is bounded above by  $\bar{m} < \infty$ , which will be validated in Theorem 3.5. Let  $\mathcal{C}[0, \bar{m}]$  denote the set of continuous and increasing functions on  $[0, \bar{m}]$ , and let  $\mathcal{V}[0, \bar{m}]$  denote the subset of  $\mathcal{C}[0, \bar{m}]$  that contains all concave functions. Taking an arbitrary ex ante value function  $V \in \mathcal{V}[0, \bar{m}]$ , we use subsection 3.1 to characterize a worker's problem. Second, with the same function  $V$ , we use subsection 3.2 to characterize a buyer's problem. Third, in subsection 3.3, we characterize an individual's lottery choice and obtain an update of the ex ante value function, denoted as  $TV$ . We prove that  $T$  is a monotone contraction mapping on  $\mathcal{V}[0, \bar{m}]$ , and so there is a unique fixed point for the ex ante value function. Finally, we verify in Theorem 3.5 that an individual's real balance is indeed bounded above by  $\bar{m} < \infty$ .

### 3.1. A worker's value and policy functions

Let  $\bar{m}$  be a sufficiently large upper bound on individuals' real balances and  $V$  any arbitrary function in  $\mathcal{V}[0, \bar{m}]$ . Given  $V$ , the worker's problem, (2.1), generates the worker's value function  $W(m)$ , the policy function of labor supply  $\ell^*(m)$ , and the policy function of the end-of-period balance  $y^*(m) = m + \ell^*(m)$ . We have the following lemma (see Appendix A for a proof):

**Lemma 3.1.** *For any  $m \in [0, \bar{m}]$  and  $V \in \mathcal{V}[0, \bar{m}]$ , the following properties hold:*

- (i)  $W \in \mathcal{V}[0, \bar{m}]$ , i.e.,  $W$  is continuous, increasing and concave on  $[0, \bar{m}]$ ;
- (ii)  $\ell^*(m)$  and  $y^*(m)$  are single-valued and continuous, with  $\ell^*(m)$  being decreasing and  $y^*(m)$  strictly increasing;
- (iii) For all  $m$  such that  $\ell^*(m) > 0$ ,  $W'(m)$  and  $V'(y^*(m))$  exist and satisfy:

$$W'(m) = \beta V'(m + \ell^*(m)) = h'(\ell^*(m)). \quad (3.1)$$

*The first equality is the envelope condition and the second equality the first-order condition.*

A worker's value function is continuous, increasing and concave in the worker's real balance because the ex ante value function has these properties. Optimal labor supply is decreasing in the worker's balance  $m$  because the marginal value of money is diminishing and the marginal disutility of labor is increasing. The end-of-period balance is strictly increasing in  $m$  because money has a strictly positive value. To establish differentiability of  $W$  in (iii), we use the standard approach in dynamic programming (see Stokey et al., 1989, p85). That is, we first show that the objective function in (2.1) is concave in  $(\ell, m)$  jointly, and then use the result in Benveniste and Scheinkman (1979) to show that  $W$  is differentiable whenever the optimal choice  $\ell^*(m)$  is interior.<sup>11</sup>

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<sup>11</sup>The choice  $\ell = 1$  is never optimal, because the marginal disutility of labor at this choice is infinite. On the other hand, if  $m$  is so high that  $\ell^*(m) = 0$ , then the individual should have entered the goods market as a buyer rather than the labor market as a worker.

The second equation in (3.1) requires an explanation, since it implies that the derivative  $V'(y^*(m))$  exists even though we have not assumed that  $V$  is differentiable. To prove that  $V'(y^*(m))$  exists, we transform a worker's problem into one where the choice variable is  $y$ :

$$W(m) = \max_{y \geq m} [\beta V(y) - h(y - m)]. \quad (3.2)$$

Concavity of  $V$  implies that  $V$  is differentiable almost everywhere and that the one-sided derivatives of  $V$  exist (see Royden, 1988, pp113-114). Then, we prove that optimal level of  $y$  is always located at a place where  $V$  is differentiable. Specifically, we use a generalized version of the envelope theorem to show that both of the one-sided derivatives of  $\beta V(y^*(m))$  are equal to those of  $W(m)$  (See Appendix A). Because  $W'(m)$  exists, so does  $\beta V'(y^*(m))$ .

Lemma 3.1 holds for all  $m \geq 0$ . Of particular interest is the case  $m = 0$ . For a worker with  $m = 0$ , denote the optimal end-of-period balance as  $\hat{m} = y^*(0) = \ell^*(0)$ . This worker's value function is  $W(0) = \beta V(\hat{m}) - h(\hat{m})$ . Lemma 3.1 implies that

$$V'(\hat{m}) = \frac{1}{\beta} h'(\hat{m}) = \frac{1}{\beta} W'(0). \quad (3.3)$$

### 3.2. A buyer's value and policy functions

We now analyze a buyer's problem (2.2), given any arbitrary ex ante value function  $V \in \mathcal{V}[0, \bar{m}]$ . In subsection 3.2.1, we reformulate the buyer's problem, describe the difficulty in analyzing the problem, outline our approach, and present the main results in Theorem 3.2. In subsections 3.2.2 and 3.2.3, we establish two lemmas which together constitute a proof of Theorem 3.2.

#### 3.2.1. The difficulty, our approach and main results

For convenience, we use  $(x, b)$  instead of  $(x, q)$  to represent a buyer's choices and express  $q$  as a function of  $(x, b)$ , where  $b$  is the buyer's matching probability in a submarket. Recall that  $b = \lambda(\theta(x, q))$ , that a trading post's matching probability is  $s = \rho(\theta(x, q))$ , and that  $s = \mu(b) \equiv \rho(\lambda^{-1}(b))$ . Thus, the market tightness condition (2.5) can be written as

$$s = \mu(b) = \begin{cases} \frac{k}{x-q}, & \text{if } k \leq x - q \\ 1, & \text{otherwise.} \end{cases} \quad (3.4)$$

In any submarket with  $x - q \leq k$ , the tightness is 0, and a buyer's matching probability is  $b = \mu^{-1}(1) = 0$ . In any submarket with  $x - q > k$ , the tightness is strictly positive, and a buyer's matching probability is  $b = \mu^{-1}(\frac{k}{x-q}) > 0$ . Thus, in any submarket  $(x, q)$  with positive tightness,

we can express the quantity of goods traded in a match as

$$q = Q(x, b) \equiv x - \frac{k}{\mu(b)}. \quad (3.5)$$

Note that if a buyer has a balance  $m \leq k$ , the only submarkets that the buyer can afford to visit have  $x - q \leq m \leq k$  and, hence, have zero tightness. For such a buyer, the optimal choice is  $b^*(m) = 0$ , and the value function is  $B(m) = \beta V(m)$ .

Let us focus on the non-trivial case  $m > k$ . In this case, the buyer's problem (2.2) can be transformed into the following one in which the choices are  $(x, b)$ :

$$\begin{aligned} B(m) &= \max_{(x,b)} \{ \beta V(m) + b[u(x, b) + \beta V(m - x) - \beta V(m)] \} \\ \text{s.t.} \quad &x \in [0, m], \quad b \in [0, 1], \end{aligned} \quad (3.6)$$

where  $u(x, b) = U(Q(x, b))$ .<sup>12</sup> Let  $(x^*(m), b^*(m))$  denote the buyer's policy functions for  $(x, b)$  and let  $\phi(m)$  denote the policy function for the residual balance  $(x - m)$ . Then,

$$q^*(m) \equiv Q(x^*(m), b^*(m)), \quad \phi(m) \equiv m - x^*(m). \quad (3.7)$$

The objective function in (3.6) is not concave jointly in  $(x, b)$  and  $m$ . The objective function involves the product of the buyer's trading probability,  $b$ , and the buyer's surplus of trade. Even if these terms are concave separately, the product of the two may not be concave in  $(x, b, m)$  jointly. The lack of concavity presents a major difficulty in using the standard approach in dynamic programming to analyze policy and value functions, because the approach starts with the requirement that the objective function be concave jointly in the choice and state variables (see Stokey et al., 1989). Attempts to make a buyer's objective function concave entail restrictions on the endogenous function  $V$ , which are difficult to verify as the outcome of (2.4).

To analyze a buyer's problem, we use lattice-theoretic techniques (see Topkis, 1998). We prove first that the policy functions are monotone and then that the value functions  $B(m)$  and  $V(m)$  are differentiable at real balances induced by optimal choices from any initial balance. These results allow us to validate the first-order conditions and envelope conditions. Finally, we prove that  $V$  is differentiable at all balances.<sup>13</sup> The main result of this procedure is stated below:

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<sup>12</sup>Note that for  $q \geq 0$ , the buyer's choices must satisfy  $x \geq k/\mu(b)$ . However, there is no need to add this constraint to the problem (3.6) because it is not binding. For any choices  $(x, b)$  such that  $x < k/\mu(b)$  and  $x > 0$ , the quantity of goods is  $q < 0$  and the utility of consumption is  $u(x, b) < U(0) = 0$ . In this case, the buyer's surplus from trade is  $u(x, b) + \beta V(m - x) - \beta V(m) < 0$ . The buyer can avoid this loss by choosing  $b = 0$ .

<sup>13</sup>There are other approaches that establish differentiability of the value function in the presence of a non-concave objective function. However, these approaches do not prove monotonicity of the policy functions. Moreover, they are not applicable in our model. Specifically, these approaches assume the objective function to be equi-differentiable (Milgrom and Segal, 2002) or differentiable with respect to the state variable (Clausen and Strub, 2010). In our model, the objective function in (2.2) contains both  $V(m)$  and  $V(m - x)$ , where  $x$  is a choice and  $m$  a state variable. For this objective function to satisfy either of the aforementioned assumptions, the value function  $V$  must be differentiable, which is a result to be proven.

**Theorem 3.2.** Take any arbitrary  $V \in \mathcal{V}[0, \bar{m}]$ . Then,  $B \in \mathcal{C}[0, \bar{m}]$ . If  $m \leq k$ , then  $b^*(m) = 0$  and  $B(m) = \beta V(m)$ ; if  $m > k$ , then  $B(m)$  satisfies (3.6). Consider any  $m \in [k, \bar{m}]$  such that  $b^*(m) > 0$ . The results (i)-(iii) below hold:

- (i) For each  $m$ , the optimal choices  $(x^*(m), b^*(m))$  and the implied quantities  $(q^*(m), \phi(m))$  are unique. The policy functions  $x^*(m)$ ,  $b^*(m)$ ,  $q^*(m)$  and  $\phi(m)$  are continuous and increasing.
- (ii) The optimal choice  $b^*(m)$  satisfies the first-order condition:

$$u(x, b) + b u_2(x, b) = \beta [V(m) - V(m - x)]. \quad (3.8)$$

For all  $m$  such that  $\phi(m) > 0$ ,  $\phi(m)$  satisfies the first-order condition:<sup>14</sup>

$$V'(\phi(m)) = \frac{1}{\beta} u_1(x^*(m), b^*(m)). \quad (3.9)$$

- (iii)  $B'(m)$  exists if and only if  $V'(m)$  exists, and  $B$  is strictly increasing.

Furthermore, consider any  $m < \bar{m}$  such that  $b^*(m) > 0$ . If  $B(m) = V(m)$  and if there exists a neighborhood  $O$  of  $m$  such that  $B(m') \leq V(m')$  for all  $m' \in O$ , then (iv) and (v) below hold.

These two parts also hold at  $m = \bar{m}$  if  $B'(\bar{m}) = V'(\bar{m})$ :

- (iv) The derivatives  $B'(m)$  and  $V'(m)$  exist and satisfy:

$$V'(m) = \frac{b^*(m)}{1 - \beta [1 - b^*(m)]} u_1(x^*(m), b^*(m)) = B'(m). \quad (3.10)$$

- (v) If  $\phi(m) > 0$ , then  $b^*$  and  $\phi$  are strictly increasing at  $m$ , and  $V$  is strictly concave at  $\phi(m)$ , with  $V'(\phi(m)) > V'(m)$ .

Parts (ii)-(iv) of this theorem assure that one can use the standard apparatus of optimization to analyze a buyer's optimal decisions and value function. We will establish Lemmas 3.3 and 3.4 below, which together prove Theorem 3.2. A reader who is eager to see the implications of the above theorem may want to go directly to subsection 3.3.

To appreciate the difficulties overcome in Theorem 3.2, let us compare the optimization problem in (3.6) with the one in Lagos and Rocheteau (2005) who apply lattice-theoretic techniques to examine comparative statics. Lagos and Rocheteau (2005) assume that individuals have quasi-linear preferences. In our notation, this assumption implies that the future value function is  $\beta V(m) = am$  for some  $a$  that is independent of an individual's choices and money balance. With this feature, the future value function is evidently differentiable. Also, the objective function in (3.6) becomes  $am + b[u(x, b) - ax]$  that is linear in  $m$  and separable between  $(x, b)$  and  $m$ . It is then clear that optimal choices of  $(x, b)$  are characterized by first-order conditions and these conditions

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<sup>14</sup>If  $\phi(m) = 0$ , then (3.9) is replaced with  $V'(0) \leq \frac{1}{\beta} u_1(m, b^*(m))$ .

do not involve the value function. Moreover, the objective function satisfies the requirements for applying the envelope condition, such as those given by the references in footnote 13. Thus, when preferences are quasi-linear, most of the analytical difficulties overcome in Theorem 3.2 are not present. Because these difficulties are associated with a non-linear future value function that arises from the wealth effect of money, they are intrinsic to monetary search models that do not employ such tractability assumptions as those in Shi (1997) and Lagos and Wright (2005).

### 3.2.2. A buyer's policy functions and monotonicity

Lattice-theoretic techniques (Topkis, 1998) to (3.6) build on the property that the objective function is supermodular in the choice and the state variables. Supermodularity (or complementary) of the objective function intuitively leads to increasing policy functions. As a preliminary step that also develops the intuition, we examine the functions  $Q(x, b)$  and  $u(x, b)$ . The function  $Q(x, b)$ , defined in (3.5), determines the quantity of goods sold to a buyer who has a matching probability  $b$  and spends a real balance  $x$  in a submarket with positive tightness. For all  $(x, b)$  such that  $Q(x, b) > 0$ , it is easy to verify that  $Q$  has the following properties:

$$Q_1(x, b) > 0, Q_2(x, b) < 0, Q(x, b) \text{ is (weakly) concave, and } Q_{12} = 0. \quad (3.11)$$

It is intuitive that  $Q$  strictly increases in  $x$  and strictly decreases in  $b$ . For any given matching probability, the more a buyer is willing to pay, the higher the quantity of goods he can obtain. For any given payment, a buyer who wants to have a higher matching probability must accept a lower quantity of goods. Equivalently, a firm must be compensated with the reduction in the quantity of goods for creating more trading posts to increase a buyer's matching probability. The second-order properties of  $Q$  are also intuitive. The function  $Q$  is strictly concave in  $b$  because increasing the number of trading posts has a diminishing marginal effect on increasing a buyer's matching probability. In order for a firm to create more trading posts to increase a buyer's matching probability further, the firm must be compensated with an increasingly larger reduction in the quantity of goods traded for a given  $x$ . The function  $Q$  is linear in  $x$  because the amount of labor needed to produce any quantity of goods is assumed to be a linear function of the quantity. Moreover,  $Q_{12} = 0$  because  $Q$  is separable in  $(x, b)$ .<sup>15</sup>

The function  $Q(x, b)$  is used in the objective function in (3.6) to express  $u(x, b) = U(Q(x, b))$ . Because the utility function  $U$  is strictly increasing and strictly concave, the properties of  $Q$  above

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<sup>15</sup>If the amount of labor needed to produce any quantity of goods is assumed to be a strictly convex function of the quantity, then  $Q$  is strictly concave in  $(x, b)$  and  $Q_{12} > 0$ . These features of  $Q$  will strengthen our results.

imply similar properties of  $u$  for all  $(x, b)$  such that  $Q(x, b) > 0$ :

$$u_1(x, b) > 0, u_2(x, b) < 0, u(x, b) \text{ is strictly concave, and } u_{12} > 0. \quad (3.12)$$

The second-order properties of  $u$  are stronger than those of  $Q$ . In particular,  $u(x, b)$  is strictly supermodular since  $u_{12} > 0$ . This property is intuitive. Consider  $u_1(x, b)$ , the marginal utility of spending. In a submarket where the buyer's matching probability is relatively high, the quantity of goods that the buyer obtains in a trade with any given spending is relatively low. At such low consumption, an increase in spending can increase the utility of consumption by a relatively large amount. Thus,  $u_1(x, b)$  is strictly higher in a submarket with a higher  $b$  than with a lower  $b$ .

Unfortunately, a buyer's objective function in (3.6) is not supermodular, despite the above properties of  $u(x, b)$ . To see this, let us write this objective function as  $\beta V(m) + bR(x, b, m)$ , where  $R$  is the buyer's surplus in the trade:

$$R(x, b, m) \equiv u(x, b) + \beta V(m - x) - \beta V(m). \quad (3.13)$$

Even if  $b$  and  $R(x, b, m)$  are supermodular, their product is not necessarily supermodular. Thus, we cannot apply Topkis' (1998) theorems directly to (3.6). To resolve this problem, we decompose the buyer's problem into two steps. In the first step, we fix  $b$  and characterize the optimal choice of  $x$ . For any given  $(b, m)$ , the optimal choice of  $x$  maximizes  $R(x, b, m)$ . Denote

$$\tilde{x}(b, m) = \arg \max_{x \in [0, m]} R(x, b, m), \quad \tilde{R}(b, m) = R(\tilde{x}(b, m), b, m). \quad (3.14)$$

In the second step, we characterize the optimal choice of  $b$  as

$$b^*(m) = \arg \max_{b \in [0, 1]} b \tilde{R}(b, m). \quad (3.15)$$

Much weaker properties are required to apply Topkis' theorems in the two steps than in the direct approach. In the first step, we prove that a buyer's surplus  $R(x, b, m)$ , rather than  $bR$ , is supermodular in  $(x, b, m)$ . Because a higher  $m$  enlarges the feasibility set in (3.14), supermodularity of  $R$  implies that  $\tilde{x}(b, m)$  is increasing in  $(b, m)$  and that  $\tilde{R}(b, m)$  is supermodular. In the second step, we prove that  $b\tilde{R}(b, m)$  is supermodular in  $(b, m)$ . That is,  $bR(x, b, m)$  is supermodular in  $(b, m)$  at the *particular spending level*  $x = \tilde{x}(b, m)$ , which is weaker than supermodularity of  $bR(x, b, m)$  for all  $(x, b, m)$ . Since the feasibility set in (3.15) is independent of  $m$ , supermodularity of  $b\tilde{R}(b, m)$  implies that the policy function  $b^*(m)$  is increasing. This implies that the policy function  $x^*(m) = \tilde{x}(b^*(m), m)$  is increasing. By changing the choices from  $(x, b)$  to  $(x, q)$  and to  $(m - x, b)$ , in turn, we use the same procedure to prove that  $q^*(m)$  and  $\phi(m)$  are increasing. The following lemma summarizes the results (see Appendix B for a proof):

**Lemma 3.3.** *For any  $V \in \mathcal{V}[0, \bar{m}]$ ,  $B(m) \in \mathcal{C}[0, \bar{m}]$ . If  $m \leq k$ , then  $b^*(m) = 0$  and  $B(m) = \beta V(m)$ ; if  $m > k$ ,  $B(m)$  solves (3.6). Moreover, for all  $m \in [k, \bar{m}]$  such that  $b^*(m) > 0$ , the policy functions are monotone as stated in part (i) of Theorem 3.2.*

It is intuitive that optimal choices  $(x^*(m), b^*(m))$  are unique for any given  $m$ , as stated in part (i) of Theorem 3.2. Because the matching function has diminishing marginal productivity for each input, a buyer's matching probability is strictly concave in the market tightness. This implies that the buyer must increase spending by an increasingly larger amount in order to increase the matching probability further by a given amount. Thus, for any given balance, there is a unique combination of  $(x, b)$  that is optimal for a buyer. That is, given the balance, a buyer chooses a unique submarket to enter, rather than being indifferent between different submarkets.

The policy functions  $x^*(m)$ ,  $b^*(m)$ ,  $q^*(m)$  and  $\phi(m)$  are all increasing, provided  $b^*(m) > 0$ . To explain this feature, note that the marginal value of a buyer's residual balance is diminishing, because  $V$  is concave. For a buyer with a higher balance, it is then optimal to increase current utility by entering a submarket where he has a higher matching probability and higher spending. In addition, because consumption is "normal" in both the current and the next period, it is optimal for a buyer with a higher balance to increase consumption in both periods. This requires the residual balance  $\phi(m)$  to be increasing in the buyer's current balance. Not surprisingly, the proof of supermodularity of a buyer's surplus function,  $R(x, b, m)$ , relies heavily on concavity of  $V$  and the properties of  $u(x, b)$  listed in (3.12).

In summary, buyers sort themselves into different submarkets according to the real balance. A buyer with more money chooses to enter a submarket where he will have a higher matching probability and once he is matched in the submarket, he will spend a larger amount of money, buy a larger quantity of goods and exit the trade with a higher balance.

### 3.2.3. First-order conditions, envelope theorems and value functions

The remaining parts of Theorem 3.2, (ii)-(v), describe the first-order conditions, the envelope condition and additional properties of the value functions. They are restated in the following lemma and proven in Appendix C:

**Lemma 3.4.** *Consider any  $m \in [k, \bar{m}]$  such that  $b^*(m) > 0$ . For any  $V \in \mathcal{V}[0, \bar{m}]$ , parts (ii) and (iii) of Theorem 3.2 hold. For any  $m < \bar{m}$  such that  $b^*(m) > 0$ , if  $B(m) = V(m)$  and if there exists a neighborhood  $O$  of  $m$  such that  $B(m') \leq V(m')$  for all  $m' \in O$ , then parts (iv) and (v) of Theorem 3.2 hold. These two parts also hold at  $m = \bar{m}$  if  $B'(\bar{m}) = V'(\bar{m})$ .*

Part (ii) of Theorem 3.2 states that optimal choices  $b^*$  and  $x^*$  satisfy the first-order conditions. In the first-order condition of  $b^*$ , (3.8), the left-hand side is the marginal benefit of increasing the trading probability,  $b$ . The right-hand side of (3.8) is the buyer's opportunity cost of a trade, represented by the reduction in the future value function resulting from a lower future balance. Similarly, the first-order condition of  $x^*$ , (3.9), requires that the marginal cost of increasing spending, represented by the marginal value of the residual balance  $\phi$ , should be equal to the marginal utility of consumption brought about by higher spending.

The first-order condition of  $b^*$  holds regardless of whether or not  $V$  is differentiable, because  $b$  does not appear in  $V$ . In contrast, the choice  $x$  appears in  $V$  through the residual balance. Thus, the first-order condition of the optimal choice  $x^*$ , (3.9), implies that the derivative  $V'(\phi(m))$  exists. That is, it is optimal for a buyer to choose spending in such a way to steer away from residual balances at which  $V$  is not differentiable. This result is established in a way similar to the existence of  $V'(y^*(m))$  in Lemma 3.1. Once  $V'(\phi(m))$  exists,  $V(m)$  is the only function on the right-hand side of the Bellman equation for  $B(m)$  whose differentiability has not been proven. It is then not surprising that  $B'(m)$  exists if and only if  $V'(m)$  does, as stated in part (iii) of Theorem 3.2. Moreover,  $B$  is strictly increasing because utility is strictly increasing in consumption.

Part (iv) of Theorem 3.2 states that  $V(m)$  is differentiable and that the envelope condition holds for a buyer's problem, for any  $m < \bar{m}$  with  $B(m) = V(m)$  and with the described neighborhood  $O$ . For such  $m$ ,  $V'(m)$  and  $B'(m)$  are both equal to the discounted marginal utility of consumption given by the expression in the middle of (3.10). To explain, note that in the neighborhood  $O$ , the one-sided derivatives of  $B$  and  $V$  satisfy  $B'(m^+) \leq V'(m^+)$  and  $B'(m^-) \geq V'(m^-)$ . For  $B'(m^+) \leq V'(m^+)$ , the discounted marginal utility of consumption cannot exceed the marginal value of retaining a unit of the balance,  $V'(m^+)$ ; for  $B'(m^-) \geq V'(m^-)$ , the discounted marginal utility of consumption cannot fall below the marginal value of spending a unit of balance,  $V'(m^-)$ . These requirements imply  $V'(m^+) \geq V'(m^-)$ . Because concavity of  $V$  implies  $V'(m^+) \leq V'(m^-)$ , then  $V'(m^+) = V'(m^-)$ ; i.e.,  $V'(m)$  exists.<sup>16</sup>

Part (v) of Theorem 3.2 describes additional properties. First, for any  $m$  with  $B(m) = V(m)$ , a buyer with the balance  $m$  does not have the need to use a lottery. This implies that the marginal value of the real balance is strictly decreasing at  $m$  and so, for any given trading probability, the buyer's surplus of trade is strictly increasing in  $m$  locally. If such a buyer has additional money, he prefers to enter a submarket with a strictly higher trading probability and spend more in order

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<sup>16</sup>The neighborhood  $O$  always exists if  $V$  is the *equilibrium* function. The neighborhood is required in Theorem 3.2 because the theorem takes  $V$  as an arbitrary function in  $\mathcal{V}[0, \bar{m}]$ . Also, the neighborhood  $O$  may not exist around the arbitrary upper bound  $\bar{m}$ , because we have not characterized  $B$  and  $V$  for  $m > \bar{m}$ . This is why the additional condition  $B'(\bar{m}) = V'(\bar{m})$  is needed for part (iv) of Theorem 3.2 to hold at  $m = \bar{m}$ .



to capture the higher surplus of trade. That is,  $b^*(m)$  and  $x^*(m)$  are strictly increasing at such  $m$ . Second, since future consumption is a normal good, it is optimal for the buyer to keep part of this additional money as the residual balance. That is,  $\phi(m)$  is also strictly increasing at such  $m$ . Third, the ex ante value function must be strictly concave at  $\phi(m) > 0$ : if  $V$  is linear at  $\phi(m)$ , the buyer should have spent more because the marginal cost of doing so is locally constant.

### 3.3. Lotteries and the equilibrium ex ante value function

We have characterized a worker's and a buyer's optimal decisions, taking the ex ante value function as any arbitrary  $V \in \mathcal{V}[0, \bar{m}]$ , i.e., any continuous, increasing and concave function on  $[0, \bar{m}]$ . Part (i) of Lemma 3.1 states that a worker's problem (2.1) defines a mapping  $T_W : \mathcal{V}[0, \bar{m}] \rightarrow \mathcal{V}[0, \bar{m}]$  that maps  $V \in \mathcal{V}[0, \bar{m}]$  into a worker's value function  $W \in \mathcal{V}[0, \bar{m}]$ . Theorem 3.2 implies that a buyer's problem (2.2) defines a mapping  $T_B : \mathcal{V}[0, \bar{m}] \rightarrow \mathcal{C}[0, \bar{m}]$  that maps  $V \in \mathcal{V}[0, \bar{m}]$  into a buyer's value function  $B \in \mathcal{C}[0, \bar{m}]$ . In the equilibrium, the ex ante value function must satisfy (2.4). If we substitute  $W = T_W V$  and  $B = T_B V$  into (2.3) to obtain  $\tilde{V}$ , then the right-hand side of (2.4) defines a mapping  $T$  on  $V$ , and we can write (2.4) as  $V(m) = TV(m)$ . That is, the equilibrium ex ante value function is a fixed point of  $T$ . In this subsection, we show that  $T$  maps  $\mathcal{V}[0, \bar{m}]$  into  $\mathcal{V}[0, \bar{m}]$  and that it has a unique fixed point. Moreover, we verify that there indeed exists a finite upper bound  $\bar{m}$  on individuals' real balances in the equilibrium.

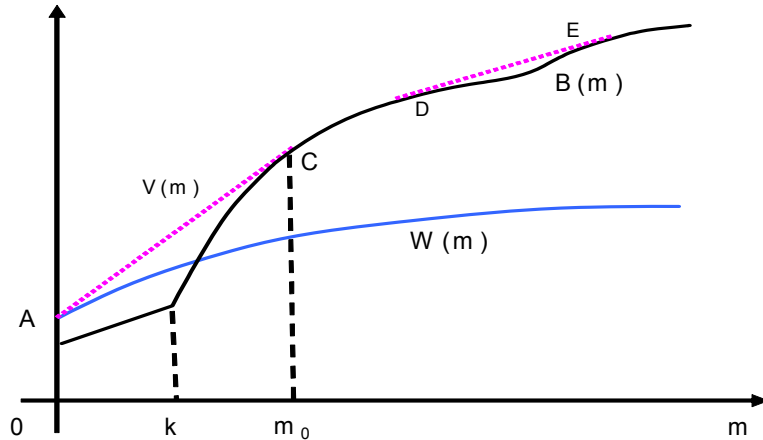


Figure 1. Lotteries and the ex ante value function

The functional equation (2.4) involves maximization over the choice of lotteries. This choice is necessary for  $T$  to preserve concavity of the ex ante value function  $V$  that has been used in the analysis above. If there were no lotteries, the ex ante value function would be  $\tilde{V}$  defined by (2.3), instead of  $V$ . The function  $\tilde{V}$  can fail to be concave for two reasons. One is that a buyer's value function,  $B(m)$ , may be non-concave. If this happens when  $B(m) > W(m)$ , then  $\tilde{V}(m)$  is

equal to  $B(m)$  and, hence, is non-concave. In this case, there is a gain to individuals to play a fair lottery to convexify the feasibility set of values. Depicted in Figure 1, this *lottery for high balances* makes  $V$  the dashed line connecting points D and E. The second cause of non-concavity of  $\tilde{V}$  is the maximum operator in (2.3). It is intuitive that  $B(m) > W(m)$  if  $m$  is high and  $B(m) < W(m)$  if  $m$  is low. When the real balance is close to the level at which  $B(m) = W(m)$ , the maximum of  $B(m)$  and  $W(m)$  is convex, and so there is a gain from playing a fair lottery. In fact, all individuals whose real balances are below this critical level gain from a lottery. Depicted in Figure 1, this *lottery for low balances* makes  $V$  the dashed line connecting points A and C.

In the lottery for low balances, the low prize is  $z_1^* = 0$ , the high prize is  $z_2^* = m_0$ , and the probability of winning the high prize is  $\pi_2^*(m) = m/m_0$ . The high prize is determined as

$$m_0 = \arg \max_{z \geq m} \left[ \frac{m}{z} \tilde{V}(z) + \left(1 - \frac{m}{z}\right) \tilde{V}(0) \right]. \quad (3.16)$$

It is clear that  $m_0$  is independent of the individual's balance  $m$ , provided  $m \leq m_0$ .

The following theorem states existence, uniqueness and other properties of the equilibrium value functions as well as the properties of the upper bound  $\bar{m}$  (see Appendix D for a proof):

**Theorem 3.5.** (i)  $T$  is a self-map on  $\mathcal{V}[0, \bar{m}]$  and has a unique fixed point  $V$ .

(ii)  $V(m) > W(m) > 0$  for all  $m > 0$ ;  $V(0) = W(0) > 0$ , and  $W(m) \geq B(m)$  for all  $m \in [0, k]$ .

(iii) There exists  $m_0 \in (k, \bar{m}]$  such that an individual with  $m < m_0$  will play the lottery with the prize  $m_0$ , which satisfies  $V(m_0) = B(m_0)$ ,  $b^*(m_0) > 0$  and  $\phi(m_0) = 0$ . Moreover, if  $m_0 < \bar{m}$ , then (3.10) holds for  $m = m_0$ , and  $V'(m_0) = B'(m_0) > 0$ .

(iv)  $V'(m) > 0$  exists for all  $m \in [0, \bar{m}]$ ;  $B'(m)$  exists for all  $m \in [k, \bar{m})$  such that  $b^*(m) > 0$ .

(v) There exists  $\bar{m} < \infty$  such that individuals' balances satisfy  $m \leq \bar{m}$  in equilibrium. Moreover,  $\bar{m}$  satisfies  $\bar{m} = \hat{z}_2 = z_2^*(\hat{m})$ ,  $B(\bar{m}) = V(\bar{m})$  and  $B'(\bar{m}) = V'(\bar{m})$ .

For part (i), we verify that the mappings on  $V$  defined by a worker's problem, (2.1), and a buyer's problem, (2.2), are monotone and feature discounting with the factor  $\beta$ . As a result,  $T$  is a monotone contraction mapping that maps continuous, increasing and concave functions into continuous and increasing functions. In addition, since  $TV$  is generated by the optimal choice of a two-point lottery, it is a concave function (see Lemma F.1, Menzio and Shi, 2010). Thus,  $T$  is a monotone contraction mapping on  $\mathcal{V}[0, \bar{m}]$  and has a unique fixed point.

Part (ii) compares  $V$ ,  $W$  and  $B$ , as depicted in Figure 1. Part (iii) formally characterizes the lottery for low balances. In particular, after winning the prize  $m_0$ , an individual strictly prefers to be a buyer and to spend the entire balance  $m_0$  in one trade. Note that this part implies that

all workers hold zero balance prior to work in the equilibrium. Part (iv) states that the ex ante value function is differentiable and strictly increasing for all  $m < \bar{m}$ , not just for the selected set of points described in Theorem 3.2. To explain, suppose that  $V$  is not differentiable at some  $m \in (m_0, \bar{m})$ , then  $V$  is strictly concave at  $m$  and hence  $V(m) = B(m)$ , in which case part (iv) of Theorem 3.2 implies the contradiction that  $V$  is differentiable at  $m$ . Thus,  $V$  is differentiable for all  $m \in (m_0, \bar{m})$ . Also,  $V$  is differentiable for all  $m \in [0, m_0)$  because  $V$  is a linear function in this interval. Furthermore, if  $m_0 < \bar{m}$ , then  $V$  is differentiable at  $m_0$  as a result of part (iv) of Theorem 3.2 and the fact  $B(m_0) = V(m_0)$ .

Finally, part (v) states that individuals' real balances are endogenously bounded above  $\bar{m} < \infty$ , which is equal to the high prize of the lottery at  $\hat{m}$ . Recall that  $\hat{m}$  is the wage income of a worker who has zero balance prior to work. Moreover, the upper bound satisfies  $B(\bar{m}) = V(\bar{m})$  and  $B'(\bar{m}) = V'(\bar{m})$ , and so we can eliminate the qualifications "if  $m < \bar{m}$ " and "if  $m_0 < \bar{m}$ " in various parts of Theorems 3.2 and 3.5. It is intuitive that individuals' real balances are finite in the equilibrium. Because the marginal utility of consumption is diminishing, the marginal value of the real balance is diminishing. In contrast, the marginal cost of labor needed to obtain money is strictly increasing. Thus, if an individual has a sufficiently large balance, he strictly prefers spending some money rather to working and accumulating even more money. This force endogenously puts an upper bound on the real balance in the equilibrium.

Although it is intuitive that equilibrium balances are bounded above, proving the result is not simple. In order to demonstrate the optimality of the bounded balance, we need to compare the value function over all possible balances including, in principle, an infinite balance. However, allowing for an infinite balance makes the choice set non-compact which, in turn, makes it difficult to characterize optimal decisions. Instead, we determine the value functions under any arbitrarily fixed and finite upper bound  $\bar{m}$ , and then vary  $\bar{m}$  to prove that individuals never chooses to hold an infinite balance. As the first step, we recognize that a worker's wage income in a period is a finite number  $\hat{m} = \ell^*(0) \leq 1$ . If the high prize of the lottery at  $\hat{m}$  is finite, then  $\bar{m} = z_2^*(\hat{m})$  is the endogenous upper bound. Thus, the only potential case in which equilibrium balances are unbounded is when  $z_2^*(\hat{m})$  is infinite, in which case the lottery at  $\hat{m}$  is not well-defined for endogenously determined  $\bar{m}$ . For this unbounded case to occur, a buyer's value function,  $B(m)$ , must be strictly increasing and (weakly) convex for all large enough  $m$ . In addition, the marginal value of the balance near  $\bar{m}$  must be increasing. But these two conditions are inconsistent with the diminishing marginal utility of consumption, because the value of money is derived ultimately from the utility of consumption that money buys. In Appendix D, we formalize these

two conditions and the proof.<sup>17</sup>

## 4. Monetary Equilibrium

In this section we characterize the spending pattern, prove existence and uniqueness of the monetary steady state, and examine the steady-state distribution of real balances.

### 4.1. Equilibrium pattern of spending

Let us begin with some features of optimal choices established in section 3. First, all workers go to work with zero balance and obtain wage income  $\hat{m}$ . Second, an individual with  $m < m_0$  plays a lottery before going to the market, with the prizes  $z_1^*(m) = 0$  and  $z_2^*(m) = m_0$ . An individual with  $m > m_0$  may or may not play a lottery. Third, buyers sort into submarkets according to the real balance. A buyer with  $m$  enters the submarket that has the matching probability  $b^*(m)$  for the buyer and the terms of trade  $(x^*(m), q^*(m))$ . The residual balance after trade is  $\phi(m) = m - x^*(m)$ . The functions  $b^*(m)$ ,  $x^*(m)$ ,  $q^*(m)$  and  $\phi(m)$  are all increasing.

Denote  $\phi^0(m) = m$  and  $\phi^{i+1}(m) = \phi(\phi^i(m))$  for  $i = 0, 1, 2, \dots$ . For any arbitrary  $m \geq m_0$ , let  $n(m)$  be the number of purchases that a buyer with  $m$  can make before his balance falls below  $m_0$ , i.e.,  $\phi^{n(m)-1}(m) \geq m_0 > \phi^{n(m)}(m)$ . Also, denote  $\hat{n} = n(\hat{m})$ ,  $\hat{z}_j = z_j^*(\hat{m})$  and  $\hat{n}_j = n(\hat{z}_j)$ , where  $j \in \{1, 2\}$ . We prove the following lemma in Appendix E:

- Lemma 4.1.** (i) If  $\hat{m} < m_0$ , then  $\hat{z}_1 = 0$ ,  $\hat{z}_2 = m_0$ , and  $\hat{n}_2 = 1$ ;  
(ii) The only lottery that is possibly played in the steady state is the lottery at  $\hat{m}$ , with  $\hat{z}_1$  and  $\hat{z}_2$  as the prizes, and this lottery is indeed played iff  $B(\hat{m}) < V(\hat{m})$ ;  
(iii) If  $\hat{m} \geq m_0$ , then the following properties hold for  $j \in \{1, 2\}$ : (a)  $b^*(\phi^{i-1}(\hat{z}_j)) > 0$  for all  $i = 1, 2, \dots, \hat{n}_j$ ; (b)  $\phi^i(\hat{z}_j) \geq m_0$ ,  $V(\phi^i(\hat{z}_j)) = B(\phi^i(\hat{z}_j))$  and  $V$  is strictly concave at  $\phi^i(\hat{z}_j)$  for all  $i = 1, 2, \dots, \hat{n}_j - 1$ ; (c)  $\phi^{\hat{n}_j}(\hat{z}_j) = 0$ .

Part (i) describes the case where wage income is  $\hat{m} < m_0$ . In this case, an individual with  $\hat{m}$  plays a lottery and, if he wins the prize  $m_0$ , he will spend it in one trade. This part is implied by part (iii) of Theorem 3.5. Note that the case  $\hat{m} < m_0$  can occur if the disutility of labor,  $h(\ell)$ , is sufficiently convex. That is, when the marginal disutility increases rapidly with labor supply, it

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<sup>17</sup>The issue on the finite upper bound resolved here is related to, but different from, that in Zhou (1999). In Zhou's paper, money can be accumulated only in discrete units, search is undirected, and there is no lottery. The issue on the upper bound arises there from the feature that an individual may encounter a match in which he is a seller rather than a buyer, even if he already holds a high balance. In contrast, directed search in our model ensures that an individual who intends to buy never chooses to enter a match in which he could end up being a seller. The issue on the upper bound arises from the lottery instead.

is optimal for a worker to work only a small amount of time in a period, which leads to  $\hat{m} < m_0$ . However, regardless of how convex  $h$  is, a worker does not work for consecutive periods to build up the balance. Instead, after working for one period, an individual tries to get the prize  $m_0$  of the lottery and then go to buy goods. The use of the lottery with the prize  $m_0$  is a better way to smooth the cost of labor supply than working for consecutive periods.

Part (ii) of Lemma 4.1 states that the only possible lottery played in the steady state is the one at  $\hat{m}$ . If  $\hat{m} < m_0$ , the statement is true and a lottery will be played, as explained above. If  $\hat{m} \geq m_0$ , the statement is implied by part (iii) of Lemma 4.1, which we explain below. In this case, a lottery will be played only when  $B(\hat{m}) < V(\hat{m})$ .

Part (iii) of Lemma 4.1 describes a buyer's stylized purchasing cycle, starting with the prize  $\hat{z}_j$  of the lottery at  $\hat{m}$ .<sup>18</sup> The buyer will trade with positive probability every period until running out of money (part (a)), the value function will be strictly concave at the residual balance of each trade if this balance is strictly positive (part (b)), and in the last trade in the cycle, he will spend all of his money (part (c)). Because of (b) and (c), the buyer has no need for a lottery at any residual balance resulted from trade. Moreover, as the balance is reduced by each trade, the buyer chooses to go into the next submarket where the trading probability, the required spending and the quantity of goods are all lower than in the previous trade.

This stylized pattern can be derived from repeatedly applying parts (iv) and (v) of Theorem 3.2. Starting at the prize  $\hat{z}_j$  of the lottery at  $\hat{m} \geq m_0$ , we have  $B(\hat{z}_j) = V(\hat{z}_j)$ ,  $B'(\hat{z}_j) = V'(\hat{z}_j)$  and  $b^*(\hat{z}_j) \geq b^*(m_0) > 0$ . If the buyer spends the entire amount  $\hat{z}_j$ , he will become a worker next period. If the residual balance is  $\phi(\hat{z}_j) > 0$ , then all the hypotheses in part (v) of Theorem 3.2 are satisfied with  $m = \hat{z}_j$ . In this case, the ex ante value function is strictly concave at  $\phi(\hat{z}_j)$ , at which  $B = V$  and no lottery is needed. Then, the hypotheses in part (iv) of Theorem 3.2 are satisfied at  $m = \phi(\hat{z}_j)$ , which imply that  $B$  and  $V$  are differentiable at  $\phi(\hat{z}_j)$  and their derivatives are equal. Moreover, because  $V$  is linear for all  $m < m_0$ , strict concavity of  $V$  at  $\phi(\hat{z}_j)$  implies  $\phi(\hat{z}_j) \geq m_0$ . With the balance  $\phi(\hat{z}_j)$ , the buyer will have the next trade with strictly positive probability. If the residual balance is  $\phi^2(\hat{z}_j) = 0$ , the round of purchases ends. If the residual balance is strictly positive, we can repeat the above argument to conclude that, at  $m = \phi^2(\hat{z}_j)$ , the function  $V$  is strictly concave,  $B = V$ , and the two functions' derivatives are given by (3.10). Moreover,  $\phi^2(\hat{z}_j) \geq m_0$ , and the buyer has no need for a lottery at  $\phi^2(\hat{z}_j)$ . This pattern continues until the  $\hat{n}_j$ -th trade, in which the buyer spends all of the money.

The purchasing cycle above has some similarity to that in Baumol (1952) and Tobin (1956).

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<sup>18</sup>We will treat the case where a lottery is not played at  $\hat{m} > m_0$  as a degenerate lottery at  $\hat{m}$ .

These authors model the role of money in a reduced form by assuming that money is necessary for purchasing goods. There is a fixed cost of converting other assets, such as nominal bonds, into money, in addition to foregoing the higher return on those assets. Because of the fixed cost, an individual manages cash holdings as a problem of optimal inventory. That is, after each withdrawal of cash, an individual makes several consecutive purchases until cash holdings fall to a critical level, at which the individual makes another withdrawal. In our model, the value of money is derived endogenously from the fundamental features of preferences (no double coincidence of wants) and the information on individuals trading history (anonymity). In our model, a fixed cost is also important for the purchasing cycle, but it lies in the process of spending money rather than getting money. More precisely, the cost is one period of time that a buyer must use to participate in the matching during which the buyer cannot work. The induced trade-off between the matching probability and the amount of a purchase is missing in the Baumol-Tobin model. Another main feature of our model that is missing from the Baumol-Tobin model and its variations is that a buyer optimally chooses the trading probability and the price at which to spend money in each stage of a purchasing cycle. Intuitively, these choices are important for how monetary policy affects the real activity and the distribution of money.<sup>19</sup>

It is useful to keep in mind that consumption in the model is consumption of only cash goods and that labor supply is only the amount of work for obtaining such cash. When mapping the model to the data, one should not interpret the model literally as predicting that an individual either works or buys goods in each period. Instead, one can re-interpret the model so that everyone in every period works for and consumes an exogenous amount of credit goods and, in addition to such activities, an individual goes through the spending cycle on cash-goods consumption.

## 4.2. Equilibrium distribution of real balances

Let  $G(m)$  be the equilibrium measure of individuals holding balances less than or equal to  $m$  immediately after the lotteries are played in a period. The support of this distribution is a discrete set. Start with a worker who has just earned wage income  $\hat{m}$ . If  $\hat{m} \geq m_0$ , the individual may or may not play a lottery whose prize is either  $\hat{z}_1$  or  $\hat{z}_2$ . After receiving the prize  $\hat{z}_j$ , an individual will go through a purchasing cycle which results in a sequence of real balances,  $\{\phi^i(\hat{z}_j)\}_{i=0}^{\hat{n}_j-1}$ . In this case, the support of  $G$  is  $\{\phi^i(\hat{z}_1)\}_{i=0}^{\hat{n}_1-1} \cup \{\phi^i(\hat{z}_2)\}_{i=0}^{\hat{n}_2-1} \cup \{0\}$ . If  $\hat{m} < m_0$ , the individual with  $\hat{m}$  plays a lottery whose prizes are  $m_0$  and 0. In this case, all buyers in the goods market hold the same balance  $m_0$ , which will be spent in one trade, and so the support of  $G$  is

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<sup>19</sup>Baumol (1952) and Tobin (1956) assume an exogenous flow of income and an equal amount of spending in each period of a purchasing cycle. Jovanovic (1982) endogenizes the flow of income in the Baumol-Tobin model but keeps the assumption that consumption is constant within a spending cycle.

$\{m_0, 0\}$ . Denote the corresponding frequency function as  $g$ .

It is straightforward to calculate the steady-state distribution of real balances. In the steady state, the measure of individuals who hold each balance in the support of  $G$  should be constant over time. If  $\hat{m} \geq m_0$  (i.e.,  $\hat{n}_2 \geq 1$ ), we can express this requirement as follows:

$$0 = g(0) \hat{\pi}_j - b^*(\hat{z}_j)g(\hat{z}_j), \quad j = 1, 2; \quad (4.1)$$

$$0 = b^*(\phi^{i-1}(\hat{z}_j))g(\phi^{i-1}(\hat{z}_j)) - b^*(\phi^i(\hat{z}_j))g(\phi^i(\hat{z}_j)) \\ \text{for } 1 \leq i \leq \hat{n}_j - 1 \text{ and } j = 1, 2; \quad (4.2)$$

$$g(0) = \sum_{j=1,2} b^*(\phi^{\hat{n}_j-1}(\hat{z}_j))g(\phi^{\hat{n}_j-1}(\hat{z}_j)), \quad (4.3)$$

where  $\hat{\pi}_j = \pi_j^*(\hat{m})$  for  $j \in \{1, 2\}$ . Equations (4.1) and (4.2) set the change in the measure of individuals who hold the balance  $\phi^i(\hat{z}_j)$  to zero, where  $i \in \{0, 1, 2, \dots, \hat{n}_j - 1\}$ . The two equations are similar, and so we explain only (4.2). The inflow of individuals into the balance  $\phi^i(\hat{z}_j)$  consists of the buyers with the balance  $\phi^{i-1}(\hat{z}_j)$  who successfully trade in the current period, and the outflow consists of the buyers with the balance  $\phi^i(\hat{z}_j)$  who successfully trade in the current period. Finally, (4.3) sets the change in the measure of individuals who hold no money to zero. In any period, the individuals who have no money are the workers. Since every worker obtains a balance  $\hat{m}$  by working for one period, the size of the outflow from the group is  $g(0)$ . The inflow comes from the buyers who are in the last period of their purchasing cycle and who successfully trade in the current period, as given by the right-hand side of (4.3).

In the case  $\hat{m} > m_0$ , (4.1) – (4.3) solve for the steady-state distribution as

$$\left. \begin{aligned} g(\phi^i(\hat{z}_j)) &= \frac{g(0)\hat{\pi}_j}{b^*(\phi^i(\hat{z}_j))} \text{ for } j = 1, 2, \text{ and } 0 \leq i \leq \hat{n}_j - 1; \\ g(0) &= \left[ 1 + \sum_{j=1,2} \sum_{i=0}^{\hat{n}_j-1} \frac{\hat{\pi}_j}{b^*(\phi^i(\hat{z}_j))} \right]^{-1}. \end{aligned} \right\} \quad (4.4)$$

The formula (4.4) is also valid in the case  $\hat{m} < m_0$ . In this case,  $\hat{z}_1 = 0$  and  $\hat{n}_2 = 1$  in (4.4), and so the steady-state distribution is  $g(m_0) = 1 - g(0)$  and  $g(0) = b^*(m_0)/[b^*(m_0) + \pi_2^*(m_0)]$ .

### 4.3. Existence and uniqueness of a monetary steady state

In section 3, we have characterized individuals' policy and value functions, which are independent of the nominal wage rate  $\omega$ . The market tightness function  $\theta$  is solved by (2.5), which is independent of  $\omega$ . Moreover, given the policy functions, (4.4) solves the steady-state distribution

of real balances independently of  $\omega$ . Thus, for a monetary steady state to exist, it suffices to solve for  $\omega$  by requirement (vi) of Definition 2.1. This requirement yields:

$$\omega = \left[ \sum_{j=1,2} \sum_{i=0}^{\hat{n}_j-1} \phi^i(\hat{z}_j) g(\phi^i(\hat{z}_j)) \right]^{-1}. \quad (4.5)$$

Because all of the elements on the right-hand side of (4.5) have been solved independently of  $\omega$ , the formula determines a unique, finite value of  $\omega$  in the steady state. We summarize this result and other properties of the steady state below (see Appendix F for a proof):

**Theorem 4.2.** *A monetary steady state must be block recursive and, moreover, it is unique. Money is neutral in the steady state. The distribution of buyers over real balances is degenerate if  $\beta \leq \beta_0$ , where  $\beta_0 > 0$  is defined in Appendix F. On the other hand, if  $\beta$  is sufficiently close to one, the distribution of buyers over real balances is non-degenerate if and only if*

$$m_c > m_0(m_c) = q_0(m_c) + \frac{k}{\mu(b_0(m_c))}, \quad (4.6)$$

where  $m_c$ ,  $q_0(m)$ , and  $b_0(m)$  are defined in Appendix F. Moreover, for each  $j = 1, 2$ , the distribution satisfies  $g(\phi^i(\hat{z}_j)) > g(\phi^{i-1}(\hat{z}_j))$  for all  $i = 1, 2, \dots, \hat{n}_j - 1$ .

There is no monetary steady state other than the block recursive one. This is true because the aggregate state including the distribution of money is constant in the steady state. One can describe a non-block recursive equilibrium by including such constant aggregate state variables as additional arguments in the value and policy functions. In the steady state, however, those functions must satisfy the same set of functional equations as described in section 3. Because those functional equations have a unique fixed point, as proven in Theorem 3.5, the value and policy functions are uniquely determined and are the same as those in the unique, block-recursive steady state. So are the distribution of money and the value of  $\omega$ . Non-block recursive monetary equilibria do not exist in the steady state.<sup>20</sup>

Money is neutral in the long run; i.e., a one-time change in the nominal stock of money has no effect on real variables in the steady state. This is intuitive in our model. The real balance,  $m$ , the quantity of money traded in a match,  $x^*$ , and the residual balance after a trade,  $\phi$ , are all measured in units of labor. They are given by the policy functions that are independent of the nominal stock. Similarly, a one-time change in the nominal stock of money does not affect

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<sup>20</sup>Outside the steady state, the dynamics of the aggregate state can affect the functional equations for the value functions, and so non-block recursive equilibria may exist.



the quantity of goods traded, labor supply and the distribution of individuals. However, money is not neutral in the short run in our model, as we will remark in section 6.

The distribution of buyers over real balances may or may not be degenerate in the steady state. The distribution is degenerate if  $\hat{m} \leq m_0$ , which occurs when individuals are sufficiently impatient in the sense  $\beta \leq \beta_0$ . In this case, an individual without money works for one period to obtain the balance  $\hat{m}$  and then plays the lottery whose prizes are  $m_0$  and 0. Thus, all buyers hold the same balance,  $m_0$ , and spend all the money whenever they have a match. An individual alternates stochastically between being a buyer with a balance  $m_0$  and a worker with no money, as determined by the lottery and the matching outcome. Thus, when  $\beta \leq \beta_0$ , our model endogenously generates the pattern of money holdings that was assumed in earlier models with indivisible money (e.g., Shi, 1995, and Trejos and Wright, 1995). Even in this case, however, our model does not share the result of those models that a one-time change in the money stock affects real activities. Instead, money is neutral in the steady state here.

It is intuitive that money distribution among buyers is degenerate when individuals are sufficiently impatient.<sup>21</sup> Consider a buyer with the highest equilibrium balance,  $\hat{z}_2$ . The buyer can spend this balance in one trade or spread it over several periods in a sequence of purchases. If the buyer spends the entire balance in one trade, he consumes a large amount of goods in the period. The upside of doing so is that the utility of current consumption is not discounted. The downside is that the marginal utility of large consumption is low, relative to spreading consumption over several periods. When the buyer is sufficiently impatient, the upside of spending all the money at once outweighs the downside. In fact, if  $\beta \leq \beta_0$ , the highest equilibrium balance is  $\hat{z}_2 = m_0$ . In this case, all buyers in the market hold the same balance  $m_0$ .

For money distribution to be non-degenerate among buyers, a necessary condition is that individuals are patient. However, high patience is not sufficient. Even in the limit  $\beta \rightarrow 1$ , it is still possible that optimal labor supply is so low that wage income is  $\hat{m} \leq m_0$ , in which case all buyers hold the same balance  $m_0$ . The additional condition, (4.6), is needed for  $\hat{m} > m_0$ . In the limit  $\beta \rightarrow 1$ , this condition is also sufficient for the distribution to be non-degenerate. Unfortunately, the condition (4.6) is too complicated to be expressed explicitly in terms of model parameters, because  $m_c$ ,  $q_0(m)$  and  $b_0(m)$  are defined implicitly through some equations (see Appendix F). To illustrate the elements involved, consider:

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<sup>21</sup>The critical level  $\beta_0$  depends on other parameters of the model. In particular,  $\beta_0$  increases in the degree of convexity of the disutility function of labor supply. Thus, consistent with an earlier explanation, the case  $\hat{m} < m_0$  is more likely to occur if the disutility function of labor supply is more convex.

*Example:*  $U(q) = \frac{(q+0.1)^{1-\sigma} - (0.1)^{1-\sigma}}{1-\sigma}$ ,  $h(\ell) = 10[1 - (1 - \ell)^\eta]$ , and  $\mu(b) = 1 - b$ . A higher value of  $\sigma$  corresponds to a more concave  $U$ , and a higher value of  $\eta$  ( $< 1$ ) to a less convex  $h$ . For any given  $(\sigma, \eta)$ , we denote  $K(\sigma, \eta)$  as the critical level of  $k$  such that (4.6) is satisfied if and only if  $k < K(\sigma, \eta)$ . Figure 2.1 depicts  $K(\sigma, 0.5)$  for  $\sigma \in [1.1, 3]$  and Figure 2.2 depicts  $K(2, \eta)$  for  $\eta \in [0.1, 0.9]$ . The function  $K(\sigma, 0.5)$  is increasing in  $\sigma$ . This means that (4.6) is more easily satisfied for any given  $(k, \eta)$  when the utility function of consumption is more concave. Also,  $K(2, \eta)$  is increasing in  $\eta$ . This means that (4.6) is more easily satisfied for any given  $(k, \eta)$  when the disutility function of labor supply is less convex.

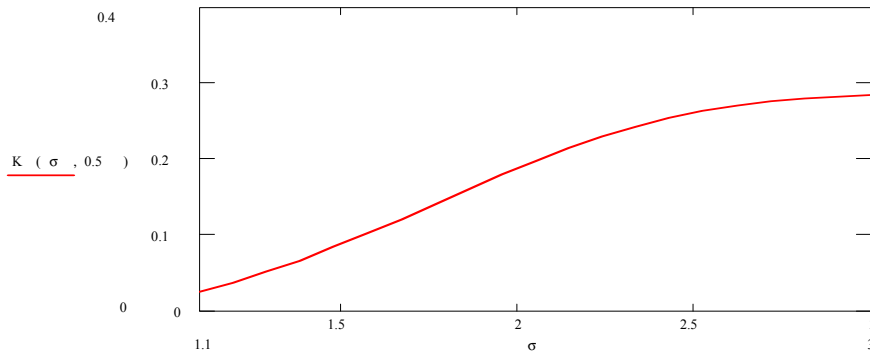


Figure 2.1

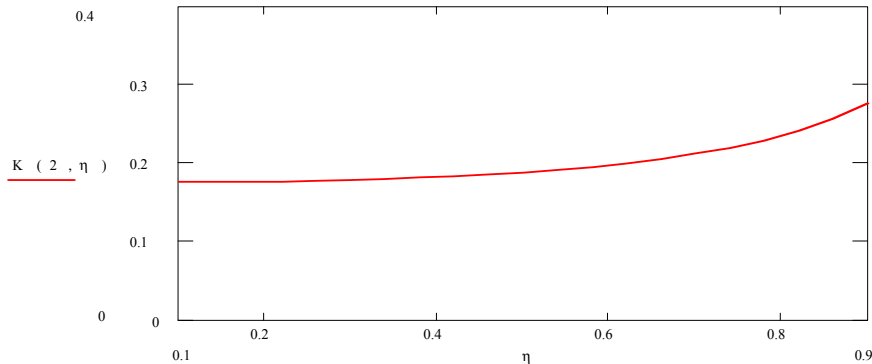


Figure 2.2

The above example illustrates the elements, in addition to high patience, that make it optimal for a buyer to spread purchases in several periods. First, the cost of creating a trading post cannot be too high. If a trading post is very costly to create, the number of trading posts in each submarket is small in the equilibrium and, hence, the matching probability is low for a buyer. Then it is optimal to spend all the money in one trade because, if the buyer keeps a positive residual balance, he will find it difficult to get a match in the future to spend the money. Second, the utility function of consumption needs to be sufficiently concave. Intuitively, a more

concave utility function increases a buyer's incentive to smooth consumption over time by making a sequence of relatively small purchases rather than one large purchase. Third, the disutility of labor supply cannot be very convex. If the marginal cost of labor supply increases very quickly, the optimal choice is to work for a relatively small balance in a period, play the lottery, spend all the prize in one trade, and work again.<sup>22</sup>

Theorem 4.2 also describes the shape of the steady-state distribution. First, consider the case where an individual with the balance  $\hat{m}$  has no need for a lottery. In this case, the frequency function of the distribution,  $g$ , is strictly decreasing in real balances among buyers. This is an intuitive implication of buyers' optimal choices described in Theorem 3.2. Because buyers with more money choose to trade with a relatively high probability, they exit quickly from the high balance into a lower balance and, hence, a relatively small number of buyers are left holding a high balance in the steady state. The measure of buyers increases as their real balances strictly decrease in the purchasing cycle. Next, consider the case where an individual with the balance  $\hat{m}$  has the need for a lottery. From each prize of the lottery,  $\hat{z}_j$  ( $j = 1, 2$ ), a buyer's balance in a purchasing cycle follows the sequence  $\{\phi^i(\hat{z}_j)\}_{i=0}^{\hat{n}_j-1}$ . The above feature of the distribution holds true for each of these two sequences. That is, for each  $j \in \{1, 2\}$ , the measure of buyers holding  $\phi^i(\hat{z}_j)$  increases with  $i$  and, hence, decreases with  $\phi^i(\hat{z}_j)$  for all  $i \in \{0, 1, \dots, \hat{n}_j - 1\}$ . However, with a non-degenerate lottery at  $\hat{m}$ , the overall frequency function of real balances is not necessarily monotone. For example,  $g(\phi^i(\hat{z}_1))$  may be greater than, less than or equal to  $g(\phi^i(\hat{z}_2))$  for a particular  $i$ , and the comparison between the two may vary over  $i$ .

A non-degenerate distribution of real balances has a wealth effect in the sense that a transfer of money between two sets of buyers with different balances affects the sum of the values of these buyers. Recall that a buyer's marginal value of the real balance increases strictly as the balance decreases with each purchase. That is,  $V'(\phi^i(\hat{z}_j)) > V'(\phi^{i-1}(\hat{z}_j))$  for all  $i = 0, 1, \dots, \hat{n}_j$  and  $j = 1, 2$  (see part (v) of Theorem 3.2). A transfer of money from a buyer with a relatively high balance to a buyer with a relatively low balance reduces the gap between the two buyers' marginal values of money. This transfer increases the sum of the values of the two buyers.

Let us relate our work to GZ. In a model of undirected search, GZ allow money to be accumulated in discrete units and assume that goods come as a fixed endowment to a seller. They prove that there exists a continuum of monetary steady states where money distribution has a discrete support and a decreasing density. In our model, the support of money distribution is also

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<sup>22</sup>Another element for a non-degenerate distribution is that  $\mu(b)$  should not increase very quickly with  $b$ . If  $\mu(b)$  increases very quickly with  $b$ , the amount of money required for obtaining any given quantity of goods increases quickly with  $b$ . In this case, the benefit of acquiring a large balance and going through a sequence of purchases is small relative to the cost of labor supply, and so a buyer will make only one purchase before working again.

discrete in the equilibrium and, when no lottery is played in the equilibrium, the density of the distribution is decreasing. The intuition for the decreasing density is similar in the two models, but the mechanism differs. In our model, the density is decreasing because a buyer with more money optimally chooses to visit a submarket where he can spend money more quickly. In GZ, individuals are not allowed to choose which market to visit, and the density is decreasing because a buyer with more money is more likely to accept an offer. There are other differences between our model and GZ. First, there is a continuum of monetary steady states in GZ. This multiplicity depends on a combination of assumptions in GZ, including the assumption that a seller cannot supply more than the unit endowment of the good. Our analysis shows that the steady state is unique when individuals can trade whatever quantities of money and goods that are optimal for them. Second, money is neutral in our model but not so in GZ. Third, an individual goes through a purchasing cycle in our model but not in GZ. Finally, because GZ assume undirected search, an extension of their model to allow for fully divisible money and goods will not be block recursive and, hence, will be intractable analytically.

## 5. Economy with Money Growth

We have assumed that the money stock is constant. In this section we outline how to incorporate non-zero money growth. Consider the following two policies: At the end of each period, each individual receives a lump-sum transfer of  $\tau\omega M$  dollars and, at the beginning of the next period, pays a proportional tax on money balance at the rate  $\delta$ . Under these policies, the aggregate stock of money evolves according to  $M_{+1} = (1 - \delta)(M + \tau\omega M)$  and the (gross) rate of money growth is  $M_{+1}/M = (1 - \delta)(1 + \tau\omega)$ . Assume that the two policies keep the money growth rate constant. Moreover, we assume  $\tau \geq 0$  so that the lump-sum transfer is always non-negative, but we allow  $\delta$  to be either positive or negative. Thus, this policy specification allows for negative as well as positive net money growth and, at the same time, avoids the use of lump-sum taxes which may be difficult to collect in an economy with decentralized exchange.

Let  $\gamma \equiv \omega_{+1}M_{+1}/(\omega M)$  denote the growth rate of the nominal wage rate. Consider a real balance  $m$  immediately before the monetary transfer. It is worth  $m\omega M$  dollars in the current period and, if it is carried over to the next period, it will become  $(1 - \delta)(m + \tau)\omega M$  dollars after the monetary transfer and the proportional tax. Measured in next period's labor, the real balance will be  $\frac{m + \tau}{\gamma/(1 - \delta)}$ . Thus, a worker's problem is modified from (2.1) to

$$W(m) = \max_{\ell \in [0,1]} \left[ \beta V\left(\frac{m + \ell + \tau}{\gamma/(1 - \delta)}\right) - h(\ell) \right]. \quad (5.1)$$

Similarly, a buyer's problem is modified from (2.2) to

$$B(m) = \max_{x \in [0, m], q \geq 0} \{ \lambda(\theta(x, q)) [U(q) + \beta V(\phi_b)] + [1 - \lambda(\theta(x, q))] \beta V(\phi_a) \}. \quad (5.2)$$

Here,  $\phi_a = \frac{m+\tau}{\gamma/(1-\delta)}$  is the individual's real balance in the next period if he does not purchase today, and  $\phi_b = \frac{m-x+\tau}{\gamma/(1-\delta)}$  the future balance if he purchases today. The optimal choice of a lottery still solves (2.4) and the firm's optimization problem still yields the relationship on the market tightness, (2.5). Because individuals take  $(\tau, \delta, \gamma)$  as given, the analytical characterization of individuals' value and policy functions in section 3 can be modified easily.

The steady state is defined in Definition 2.1, with the above optimization problems. In the steady state, the distribution of real balances is stationary, and so  $\omega$  is constant by (vi) in the equilibrium definition. Then,  $\gamma = M_{+1}/M$ , which is also constant. To maintain this money growth,  $\delta$  and  $\tau$  must satisfy:  $\gamma/(1-\delta) = 1 + \tau\omega$ . Note that  $\gamma$  and  $\delta$  do not affect the equilibrium separately, but rather through  $\gamma/(1-\delta)$ . Money growth is neutral if the lump-sum transfer rate  $\tau$  is fixed, since the proportional tax adjusts with money growth to maintain  $\gamma/(1-\delta) = 1 + \tau\omega$ . An implication is that if  $\tau = 0$ , the steady state with money growth is identical to the one with zero money growth. Thus, for any money growth rate, the steady state is block recursive if money growth is engineered entirely through the proportional tax.

If  $\tau > 0$ , then  $\omega$  enters the decision problems through the term  $(1 + \tau\omega)$ , and so the steady state is not block recursive. This is a general feature of all models with a non-degenerate money distribution because for any real transfer per capita,  $\tau$ , the aggregate amount of transfers depends on the average real balance. However, this failure of block recursivity is very mild here, because the distribution does not affect the steady state directly, but rather indirectly only through the one-dimensional variable  $\omega$  in the term  $(1 + \tau\omega)$ . As a result, the steady state is still manageable and its computation straightforward. First, taking a guess of  $\omega$ , we can solve the policy and value functions from the analysis in section 3. Second, from any initial distribution of real balances, we can use the policy functions to simulate the economy until the distribution converges to an ergodic distribution. Third, with the ergodic distribution, we invoke condition (vi) in Definition 2.1 to compute an update of  $\omega$ . Iterating on  $\omega$  according to this procedure solves the steady state. Note that this is an exercise of finding a fixed point for a real number, which is substantially simpler than finding a fixed point for a distribution function. The latter exercise is required in undirected search models, e.g., Molico (2006) and Chiu and Molico (2008).

## 6. Concluding Remarks

In this paper, we have constructed and analyzed a tractable search model of money where the distribution of real money balances can be non-degenerate. Search is directed in the sense that buyers know the terms of trade before visiting particular sellers. We showed that the monetary steady state is block recursive in the sense that individuals' policy functions, value functions and the market tightness function are all independent of the distribution of individuals over real balances, although the distribution affects the aggregate activity by itself. Using lattice-theoretic techniques, we characterized individuals' policy and value functions, and showed that these functions satisfy the standard conditions of optimization. We proved that a unique monetary steady state exists, provided conditions under which the steady-state distribution of buyers over real balances is non-degenerate, and analyzed the properties of this distribution.

We hope that our model provides a new starting point for studying monetary policy. Although the monetary steady state is block recursive, the non-degenerate distribution of money does matter for the aggregate real activity and welfare. This distribution serves as a channel through which monetary policy affects the real activity. In particular, a one-time increase in the lump-sum monetary transfer can have persistent real effects in the short run, despite that this policy is neutral in the long run. The short-run effects arise in two ways. First, the transfer changes the support of the distribution of real balances in the short run, because the transfer represents a larger proportional increase in the balance to a buyer with a low balance than to a buyer with a high balance. This redistribution of the purchasing power affects aggregate consumption. Second, the transfer changes the rate of return to money in the short run. After the shock, the nominal wage rate (normalized by the money stock),  $\omega$ , deviates immediately from the steady state. So do the rate of return to money,  $\omega_t/\omega_{t+1}$ , and the growth rate of the nominal wage,  $\gamma_t$ . As illustrated in section 5, these deviations of  $\omega$  and  $\gamma$  from the steady state affect individuals' optimal decisions. The real effects are persistent if the distribution of real balances or  $\omega_t/\omega_{t+1}$  returns to the steady state slowly.<sup>23</sup> Thus, our model has the potential to explain why temporary monetary shocks have persistent real effects as alluded to in the introduction. Furthermore, these real effects depend on how and where money is injected. In section 5, we have illustrated that proportional injections of money are neutral while lump-sum injections are not. Similarly, injecting money only into firms has different real effects from injecting money uniformly in the economy because they affect money distribution differently.

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<sup>23</sup>Because the monetary steady state is unique, the equilibrium converges to the original steady state after the described shock if it converges to any monetary steady state at all. Also, because not all individuals become workers at the same time and because labor income  $\hat{m}$  depends on  $\omega$ , the transition back to the steady state is asymptotic rather than being completed in finite time.

The steady-state analysis in this paper provides a long-run anchor for studying the dynamic effects of monetary policy described above. In the transition to the steady state, the equilibrium is not block recursive, but this failure of block recursivity is mild because policy and value functions depend on the distribution of money only through the one-dimensional variable  $\omega_t$ . Similar to the discussion in section 5, the computation of the dynamics is manageable and much simpler than in models of undirected search. This is true even in a stochastic version of our model, where the dynamic equilibrium can be solved using the approximation technique of Krusell and Smith (1998), i.e., by approximating the distribution with a small number of moments. In contrast, in undirected search models including Molico (2006) and Chiu and Molico (2008), the entire distribution of money affects individuals' policy and value functions directly. In that case, there is no intuitive reason why the Krusell-Smith approximation can work well, and the computation suffers from the well-known problem of large dimensionality. We will undertake the dynamic analysis in future work, which will also incorporate other elements relevant for policy analysis, such as nominal bonds and aggregate shocks.

## Appendix

### A. Proof of Lemma 3.1

Take any  $V \in \mathcal{V}[0, \bar{m}]$  as the ex ante value function appearing in a worker's maximization problem, (2.1). The objective function in (2.1) is continuous, bounded on  $[0, \bar{m}]$  and increasing in  $m$ . Then, the Theorem of the Maximum implies  $W \in \mathcal{C}[0, \bar{m}]$ , i.e., a continuous and increasing function on  $[0, \bar{m}]$ . Because the objective function  $[\beta V(m + \ell) - h(\ell)]$  is strictly concave in  $(\ell, m)$  jointly, its maximized value,  $W(m)$ , is concave in  $m$ , and the optimal choice  $\ell^*$  is unique. With uniqueness, the Theorem of the Maximum implies that the policy function  $\ell^*(m)$  is continuous (see Stokey et al., 1989, p62). The choice  $\ell = 1$  can never be optimal under the assumption  $h'(1) = \infty$ . It may be possible that the optimal choice is  $\ell^*(m) = 0$  when  $m$  is sufficiently high. In this case, it is evident that  $\ell^*(m) = 0$  is (weakly) increasing in  $m$  and  $y^*(m) = m$  is strictly increasing in  $m$ .

The remainder of this proof focuses on the case where  $\ell^*(m) > 0$ . In this case,  $y^*(m) = m + \ell^*(m) > m$ . Reformulate a worker's problem as (3.2), where the choice is the end-of-period balance  $y = m + \ell$ . The objective function in (3.2) is strictly concave in  $(y, m)$  jointly and  $h(y - m)$  is continuously differentiable in  $(y, m)$ . Thus, the result in Benveniste and Scheinkman (1979) applies (see also Stokey et al., 1989, p85). That is, for all  $m$  such that the optimal choice  $y^*(m)$  is interior,  $W(m)$  is differentiable and the derivative satisfies:

$$W'(m) = h'(y^*(m) - m) = h'(\ell^*(m)).$$

In addition, using concavity of  $W$  and strict convexity of  $h$ , we can deduce from the equation  $W'(m) = h'(\ell^*(m))$  that  $\ell^*(m)$  is decreasing in  $m$ .

Return to the original maximization problem of a worker, (2.1). Consider any  $m \in [0, \bar{m}]$  such that  $\ell^*(m) > 0$ . Because  $\ell^*(m)$  is continuous, there exists  $\varepsilon_0 > 0$  such that  $\ell^*(m \pm \varepsilon) > 0$  for all  $\varepsilon \in [0, \varepsilon_0]$ . Moreover, we can choose sufficiently small  $\varepsilon_0$  so that for any  $\varepsilon \in [0, \varepsilon_0]$ , the choice  $\ell^*(m - \varepsilon)$  is feasible to a worker who holds a balance  $m$  and the choice  $\ell^*(m)$  is feasible to a worker who holds a balance  $m - \varepsilon$ . Then, for any  $\varepsilon \in [0, \varepsilon_0]$ , the optimality of  $\ell^*$  implies:

$$\begin{aligned} W(m) &= F(\ell^*(m), m) \geq F(\ell^*(m - \varepsilon), m), \\ W(m - \varepsilon) &= F(\ell^*(m - \varepsilon), m - \varepsilon) \geq F(\ell^*(m), m - \varepsilon), \end{aligned}$$

where  $F(\ell, m)$  temporarily denotes the objective function in (2.1). Hence,

$$\frac{F(\ell^*(m - \varepsilon), m) - F(\ell^*(m - \varepsilon), m - \varepsilon)}{\varepsilon} \leq \frac{W(m) - W(m - \varepsilon)}{\varepsilon} \leq \frac{F(\ell^*(m), m) - F(\ell^*(m), m - \varepsilon)}{\varepsilon}.$$

Since  $W'(m)$  exists, taking the limit  $\varepsilon \searrow 0$  on the above relations yields  $\beta V'(y^{*-}(m)) = W'(m)$ , where  $y^{*-}(m) = m^- + \ell^*(m)$ . Note that the one-sided derivatives  $V'(y^{*-})$  and  $V'(y^{*+})$  exist



because  $V$  is a concave function (see Royden, 1988, pp113-114). Similarly, we can prove that  $\beta V'(y^{*+}(m)) = W'(m)$ , where  $y^{*+}(m) = m^+ + \ell^*(m)$ . Therefore,  $V$  is differentiable at  $y^*(m)$  and the derivative satisfies  $\beta V'(y^*(m)) = W'(m)$ , which is the first equality in (3.1). Substituting  $W'(m) = h'(\ell^*(m))$  yields the second equality in (3.1).

Finally, since  $V'$  is decreasing and  $h'$  is strictly increasing, the first-order condition  $\beta V'(y^*(m)) = h'(y^*(m) - m)$  implies that  $y^*(m)$  is strictly increasing. **QED**

## B. Proof of Lemma 3.3

Take any  $V \in \mathcal{V}[0, \bar{m}]$  as the ex ante value function appearing in a buyer's maximization problem, (2.2). Applying the Theorem of the Maximum to the problem, we conclude that  $B$  is continuous on  $[0, \bar{m}]$  and that a solution to the buyer's maximization problem exists (see Stokey et al., 1989, p62). Since  $V$  is increasing, the objective function in (2.2) is increasing in  $m$ . Since the feasibility set in the maximization problem is also increasing in  $m$ , then  $B$  is increasing, i.e.,  $B \in \mathcal{C}[0, \bar{m}]$ . As explained earlier in subsection 3.2.1,  $B$  has two segments. If  $m \leq k$ , then  $b^*(m) = 0$  and  $B(m) = \beta V(m)$ ; if  $m > k$ ,  $B(m)$  solves (3.6).

If  $b^* = 0$ , the choice of  $x$  is irrelevant for the buyer because a trade does not take place. For the remainder of the proof, we focus on the case where  $b^*(m) > 0$ . Temporarily denote  $F(x, b, m) = b R(x, b, m)$ , where the surplus function  $R$  is defined in (3.13). Optimal choices  $(x^*, b^*)$  maximize  $F(x, b, m)$ .

(1) A buyer's optimal choices are unique and the policy functions are continuous.

If  $R(x, b^*, m) < 0$ , the optimal choice is  $b^* = 0$ ; if  $R(x, b^*, m) = 0$ , then the choice  $b^* = 0$  is not dominated by other choices of  $b$ . Because we focus on  $b^* > 0$ , it suffices to examine a buyer's optimal choices when  $R(x, b, m) > 0$ . With  $b > 0$  and  $R(x, b, m) > 0$ , we can transform a buyer's maximization problem as

$$B(m) = \beta V(m) + \exp \left\{ \max_{x,b} [\ln b + \ln R(x, b, m)] \right\}.$$

The function  $(\ln b)$  is concave. Recall that  $u(x, b)$  is strictly concave in  $(x, b)$  jointly. Since  $V$  is concave, then  $V(m - x)$  is concave in  $x$ . Thus,  $R(x, b, m)$  defined in (3.13) is strictly concave in  $(x, b)$  jointly. Since the logarithmic function is strictly increasing and strictly concave, the function  $[\ln b + \ln R(x, b, m)]$  is strictly concave in  $(x, b)$  jointly. The Theorem of the Maximum implies that a buyer's optimal choices  $(x^*, b^*)$  are unique for each  $m$  and the policy functions  $(x^*(m), b^*(m))$  are continuous. So are the policy functions  $q^*(m)$  and  $\phi(m)$ .

(2) Monotonicity of the policy functions  $x^*(m)$  and  $b^*(m)$ .

Consider any  $m \in [k, \bar{m}]$  such that  $b^*(m) > 0$ . As discussed in the main text, we solve a buyer's maximization problem in two steps: first, the optimal choice of  $x$  solves the problem in (3.14) for any given  $(b, m)$ ; second, the optimal choice of  $b$  solves the problem in (3.15).

Take the first step. For any given  $(b, m)$ , the optimal choice of  $x$  maximizes  $R(x, b, m)$  and is denoted  $\tilde{x}(b, m)$  as in (3.14). Because  $u(x, b)$  is strictly concave in  $x$  and  $V$  is concave,  $R(x, b, m)$  is strictly concave in  $x$ , which implies that a unique  $\tilde{x}$  exists for any given  $(b, m)$ . We prove that  $R(x, b, m)$  is supermodular. Since the choice set of  $x$ ,  $[0, m]$ , is increasing in  $m$  and independent of  $b$ , supermodularity of  $R(x, b, m)$  and uniqueness of  $\tilde{x}$  imply that the maximizer  $\tilde{x}(b, m)$  is an increasing function of  $(b, m)$  (see Topkis, 1998, p76) and that the maximized value of  $R$  is supermodular in  $(b, m)$  (see Topkis, 1998, p70).

To prove that  $R(x, b, m)$  is supermodular, note that the feasibility set of  $(x, b, m)$  is  $\{(x, b, m) : 0 \leq x \leq m, 0 \leq b \leq 1, k \leq m \leq \bar{m}\}$ . This set is a sublattice in  $\mathbb{R}_+^3$  with the usual relation " $\geq$ ". It suffices to prove that  $R$  has increasing differences in the three pairs,  $(b, m)$ ,  $(x, b)$  and  $(x, m)$  (see Topkis, 1998, p45). Take arbitrary  $m_1, m_2, x_1, x_2, b_1$  and  $b_2$  from the feasibility set, with  $m_2 > m_1, x_2 > x_1$ , and  $b_2 > b_1$ . Because  $R$  is separable in  $b$  and  $m$ , it is clear that  $R$  has (weakly) increasing differences in  $(b, m)$ . For the differences in  $(x, b)$ , compute:

$$R(x_2, b, m) - R(x_1, b, m) = [u(x_2, b) - u(x_1, b)] + \beta [V(m - x_2) - V(m - x_1)].$$

Since  $u(x, b)$  is strictly supermodular in  $(x, b)$ , we have:

$$\begin{aligned} & [R(x_2, b_2, m) - R(x_1, b_2, m)] - [R(x_2, b_1, m) - R(x_1, b_1, m)] \\ &= [u(x_2, b_2) - u(x_1, b_2)] - [u(x_2, b_1) - u(x_1, b_1)] > 0. \end{aligned}$$

That is,  $R$  has strictly increasing differences in  $(x, b)$ . For the differences in  $(x, m)$ , we have:

$$\begin{aligned} & [R(x_2, b, m_2) - R(x_1, b, m_2)] - [R(x_2, b, m_1) - R(x_1, b, m_1)] \\ &= \beta [V(m_1 - x_1) - V(m_1 - x_2)] - \beta [V(m_2 - x_1) - V(m_2 - x_2)] \geq 0. \end{aligned}$$

The inequality follows from concavity of  $V$  (see Royden, 1988, p113) and the facts that  $m_1 - x_1 < m_2 - x_1, m_1 - x_2 < m_2 - x_2$ , and  $(m_1 - x_1) - (m_1 - x_2) = (m_2 - x_1) - (m_2 - x_2) = x_2 - x_1 > 0$ .

Thus,  $R(x, b, m)$  has increasing differences in  $(x, m)$ .

Denote  $\tilde{R}(b, m) = R(\tilde{x}(b, m), b, m)$  as in (3.14). From the above proof,  $\tilde{R}(b, m)$  is supermodular in  $(b, m)$ . Because  $R(x, b, m)$  strictly decreases in  $b$  for any given  $(x, m)$ , then  $\tilde{R}(b, m)$  is strictly decreasing in  $b$ . To examine the dependence of  $\tilde{R}(b, m)$  on  $m$ , take arbitrary  $m_1$  and  $m_2$  in  $[k, \bar{m}]$ , with  $m_2 \geq m_1$ . We have:

$$R(x, b, m_2) - R(x, b, m_1) = \beta [V(m_1) - V(m_1 - x)] - \beta [V(m_2) - V(m_2 - x)] \geq 0,$$

where the inequality follows from concavity of  $V$ . Since the above result holds for all  $(x, b)$ , then

$$\tilde{R}(b, m_1) = R(\tilde{x}(b, m_1), b, m_1) \leq R(\tilde{x}(b, m_1), b, m_2) \leq R(\tilde{x}(b, m_2), b, m_2) = \tilde{R}(b, m_2).$$

Note that for the second inequality we have used the fact that  $\tilde{x}(b, m_1)$  is feasible in the problem  $\max_{x \leq m_2} R(x, b, m_2)$ . Thus,  $\tilde{R}(b, m)$  increases in  $m$ .

Now let us take the second step, i.e., to characterize the optimal choice of  $b$ . Denote the optimal choice of  $b$  as  $b^*(m) = \arg \max_{b \in [0,1]} f(b, m)$ , where

$$f(b, m) = F(\tilde{x}(b, m), b, m) = b \tilde{R}(b, m).$$

We show that  $f$  is supermodular in  $(b, m)$ . Take arbitrary  $b_1, b_2 \in [0, 1]$ , with  $b_2 > b_1$ , and arbitrary  $m_1, m_2 \in [k, \bar{m}]$ , with  $m_2 > m_1$ . Compute:

$$\begin{aligned} & [f(b_2, m_2) - f(b_1, m_2)] - [f(b_2, m_1) - f(b_1, m_1)] \\ = & b_2 \left[ \tilde{R}(b_2, m_2) - \tilde{R}(b_1, m_2) + \tilde{R}(b_1, m_1) - \tilde{R}(b_2, m_1) \right] \\ & + (b_2 - b_1) \left[ \tilde{R}(b_1, m_2) - \tilde{R}(b_1, m_1) \right]. \end{aligned}$$

Because  $\tilde{R}(b, m)$  is supermodular in  $(b, m)$ , the first difference on the right-hand side is positive. Because  $\tilde{R}(b, m)$  is increasing in  $m$ , the second difference on the right-hand side is also positive. Thus,  $f(b, m)$  is supermodular in  $(b, m)$  on  $[0, 1] \times [k, \bar{m}]$ . Note also that the choice set for  $b$ ,  $[0, 1]$ , is independent of  $m$  and that the optimal choice  $b^*$  is unique. Thus,  $b^*(m)$  is increasing in  $m$  (see Topkis, 1998, p76). Since  $\tilde{x}(b, m)$  is increasing in  $(b, m)$ , the optimal choice of  $x$ , given by  $x^*(m) = \tilde{x}(b^*(m), m)$ , is increasing in  $m$ .

(3)  $q^*(m)$  is an increasing function.

Denote  $a = m - x + q$  and use  $(q, a)$  as a buyer's choices. Using (3.4), we can express:

$$m - x = a - q, \quad b = \mu^{-1} \left( \frac{k}{m - a} \right).$$

Because  $b \geq 0$ , the relevant domain of  $a$  is  $[0, m - k]$ . The relevant domain of  $q$  is  $[0, a]$ . A buyer chooses  $(q, a) \in [0, a] \times [0, m - k]$  to solve:

$$\max_{(q,a)} \mu^{-1} \left( \frac{k}{m - a} \right) [U(q) + \beta V(a - q) - \beta V(m)].$$

We can divide this problem into two steps: first solve  $q$  for any given  $(a, m)$  and then solve  $a$ .

For any given  $(a, m)$ , the optimal choice of  $q$ , denoted as  $\tilde{q}(a)$ , solves:

$$J(a) \equiv \max_{0 \leq q \leq a} [U(q) + \beta V(a - q)].$$

Note that  $q$  and  $J$  do not depend on  $m$  for any given  $a$ . It is easy to see that the objective function above is supermodular in  $(q, a)$ . Since the choice set,  $[0, a]$ , is increasing in  $a$  and  $\tilde{q}$  is unique, then  $\tilde{q}(a)$  and  $J(a)$  increase in  $a$  (see Topkis, 1998, p76 and p70).

The optimal choice of  $a$  is  $a^*(m) = \arg \max_{0 \leq a \leq m-k} \Delta(a, m)$ , where

$$\Delta(a, m) = \mu^{-1} \left( \frac{k}{m-a} \right) [J(a) - \beta V(m)].$$

Note that if  $J(a) < \beta V(m)$ , the buyer can choose  $a = m - k$  to obtain  $\Delta = 0$ . Thus, focus on the case where  $J(a) \geq \beta V(m)$ . Since  $\mu(b)$  is strictly decreasing in  $b$  and  $1/\mu(b)$  is strictly convex in  $b$ , the function  $\mu^{-1}(\frac{k}{m-a})$  strictly increases in  $m$ , strictly decreases in  $a$ , and is strictly supermodular in  $(a, m)$ . Thus, for arbitrary  $a_2 > a_1$  and  $m_2 > m_1 \geq k$ , we have:

$$\begin{aligned} & \Delta(a_2, m_2) - \Delta(a_1, m_2) - \Delta(a_2, m_1) + \Delta(a_1, m_1) \\ = & \left[ \mu^{-1} \left( \frac{k}{m_2 - a_2} \right) - \mu^{-1} \left( \frac{k}{m_1 - a_2} \right) \right] [J(a_2) - J(a_1)] \\ & + \left[ \mu^{-1} \left( \frac{k}{m_2 - a_1} \right) - \mu^{-1} \left( \frac{k}{m_2 - a_2} \right) \right] [\beta V(m_2) - \beta V(m_1)] \\ & + \left[ \mu^{-1} \left( \frac{k}{m_2 - a_2} \right) - \mu^{-1} \left( \frac{k}{m_2 - a_1} \right) - \mu^{-1} \left( \frac{k}{m_1 - a_2} \right) + \mu^{-1} \left( \frac{k}{m_1 - a_1} \right) \right] [J(a_1) - \beta V(m_1)]. \end{aligned}$$

The first term on the right-hand side is positive because  $J(a)$  increases in  $a$  and  $\mu^{-1}(\frac{k}{m-a})$  increases in  $m$ . The second term on the right-hand side is positive because  $\mu^{-1}(\frac{k}{m-a})$  decreases in  $a$  and  $V(m)$  increases in  $m$ . The third term on the right-hand side is strictly positive because  $\mu^{-1}(\frac{k}{m-a})$  is strictly supermodular in  $(a, m)$ . Therefore,  $\Delta(a, m)$  is strictly supermodular. Since the choice set  $[0, m - k]$  is also increasing in  $m$ , the solution  $a^*(m)$  increases in  $m$  (see Topkis, 1998, p76). Since  $\tilde{q}(a)$  increases in  $a$ , then  $q^*(m) = \tilde{q}(a^*(m))$  increases in  $m$ .

(4)  $\phi(m)$  is an increasing function.

We reformulate a buyer's problem by letting the choices be  $(\phi, a)$ , where  $a$  is defined as  $a = \phi + q$ . From the definition of  $a$  and (3.4), we can express

$$q = a - \phi, \quad b = \mu^{-1} \left( \frac{k}{m-a} \right).$$

The relevant domain of  $\phi$  is  $[0, \min\{m, a\}]$ , and of  $a$  is  $[0, m - k]$ . A buyer solves:

$$\max_{(\phi, a)} \mu^{-1} \left( \frac{k}{m-a} \right) [U(a - \phi) + \beta V(\phi) - \beta V(m)]. \quad (\text{B.1})$$

As in the above formulation where the choices are  $(q, a)$ , we can divide the maximization problem into two steps. First, for any given  $a$ , the optimal choice of  $\phi$  solves:

$$J(a) = \max_{\phi \geq 0} [U(a - \phi) + \beta V(\phi)]. \quad (\text{B.2})$$

Note that we have written the constraint on  $\phi$  as  $\phi \geq 0$ , instead of  $\phi \in [0, \min\{m, a\}]$ . The optimal choice satisfies  $\phi < m$ , because  $\phi = m$  implies  $x = 0$  which is not optimal (in the case

with  $b > 0$ ). Also,  $\phi < a$  under the assumption that  $U'(0)$  is sufficiently large. Denote the solution for  $\phi$  as  $\tilde{\phi}(a)$ . Second, the optimal choice of  $a$  solves

$$B(m) - \beta V(m) = \max_{0 \leq a \leq m-k} \mu^{-1} \left( \frac{k}{m-a} \right) [J(a) - \beta V(m)]. \quad (\text{B.3})$$

Similar to the procedure used in the above formulation of the problem where the choices are  $(q, a)$ , we can prove that  $a^*(m)$  increases in  $m$  and, hence,  $\phi^*(m)$  increases in  $m$ . **QED**

### C. Proof of Lemma 3.4

Take any  $V \in \mathcal{V}[0, \bar{m}]$  as the ex ante value function appearing in a buyer's problem and consider any arbitrary  $m \in [k, \bar{m}]$  such that  $b^*(m) > 0$ . Parts (1) - (4) below establish Lemma 3.4.

(1) The one-sided derivatives of  $B$  satisfy:

$$B'(m^+) = b^*(m)u_1(x^*(m), b^*(m)) + \beta(1 - b^*(m))V'(m^+) \quad (\text{C.1})$$

$$B'(m^-) = b^*(m)u_1(x^*(m), b^*(m)) + \beta(1 - b^*(m))V'(m^-). \quad (\text{C.2})$$

$B'(m)$  exists if and only if  $V'(m)$  exists. Moreover,  $B(m)$  is strictly increasing.

Consider the formulation of a buyer's problem, (B.1), where the choices are  $\phi$  and  $a = \phi + q$ . Let  $a$  and  $a'$  be arbitrary levels in  $[0, m - k]$ . Note that the constraint on the choice  $\phi$  is  $\phi \geq 0$ , which does not depend on  $a$ . Thus, the choice  $\tilde{\phi}(a)$  is feasible in the maximization problem with  $a'$  and the choice  $\tilde{\phi}(a')$  is feasible in the maximization problem with  $a$ . Using a proof similar to the one in Appendix A that established the existence of  $V'(y^*(m))$ , we can prove that  $J'(a^-)$  and  $J'(a^+)$  both exist and are equal to

$$J'(a) = U'(\tilde{q}(a)) > 0, \quad (\text{C.3})$$

where  $\tilde{\phi}$  and  $\tilde{q}(a) \equiv a - \tilde{\phi}(a)$  are given in part (4) of the above proof of Lemma 3.3.

Next, we prove that the objective function in (B.3) is strictly concave in  $a$  and derive the first-order condition of  $a$ . Recall that  $\tilde{q}(a)$  is an increasing function, as shown in the above proof of Lemma 3.3. This result and (C.3) together imply that  $J'(a)$  is decreasing, i.e., that  $J(a)$  is concave. Because  $J(a)$  is increasing and concave, and  $\mu^{-1}(\frac{k}{m-a})$  is strictly decreasing and strictly concave in  $a$ , it can be verified that the objective function in (B.3) is strictly concave in  $a$ . Strict concavity of the objective function implies that the optimal choice of  $a$  is unique. Also, because the objective function is differentiable in  $a$ , the optimal choice of  $a$  is given by the first-order

condition. Deriving the first-order condition, substituting  $J'(a)$  from (C.3), and substituting  $\mu^{-1}(\frac{k}{m-a^*}) = b^*(m)$ , we obtain:

$$J(a^*) - \beta V(m) + U'(\tilde{q}(a^*)) \frac{k\mu' b^*(m)}{\mu^2} \leq 0 \quad \text{and} \quad a^* \leq m - k, \quad (\text{C.4})$$

where the two inequalities hold with complementary slackness.

Now we derive (C.1) and (C.2), which clearly imply that  $B'(m)$  exists if and only if  $V'(m)$  exists. Note that  $b^*(m) > 0$  implies  $a^*(m) < m - k$ . Because  $a^*(m) < m - k$  and  $a^*(m)$  is continuous, there exists  $\varepsilon > 0$  such that  $a^*(m + \varepsilon) < m - k$  and  $a^*(m) < m - \varepsilon - k$ . Consider the neighborhood  $O(m) = (m - \varepsilon, m + \varepsilon)$ . For any  $m' \in O(m)$ , the choice  $a^*(m')$  is feasible in the problem where the balance is  $m$ , and the choice  $a^*(m)$  is feasible in the problem where the balance is  $m'$ . Applying to (B.3) a proof similar to Appendix A that established the existence of  $V'(y^*(m))$ , we can derive the formulas of  $B'(m^+)$  and  $B'(m^-)$  for any  $m$  such that  $b^*(m) > 0$ . These formulas and the first-order condition of  $a^*$ , (C.4), together yield:

$$\begin{aligned} B'(m^+) &= b^*(m) [J'(a^*) - \beta V'(m^+)] + \beta V'(m^+), \\ B'(m^-) &= b^*(m) [J'(a^*) - \beta V'(m^-)] + \beta V'(m^-). \end{aligned}$$

Again, we have used the fact that a concave function has one-sided derivatives. Substituting  $J'(a^*)$  from (C.3) and  $U'(q^*) = u_1(x^*, b^*)$  into the above equations, we obtain (C.1) and (C.2).

Finally, we prove that  $B$  is strictly increasing. Since  $V$  is concave and increasing,  $V'(m^-) \geq V'(m^+) \geq 0$ . Since  $b^* \leq 1$  and  $J'(a^*(m)) > 0$ , the above equations for  $B'(m^+)$  and  $B'(m^-)$  imply that  $B'(m^-) \geq B'(m^+) \geq b^*(m)J'(a^*) > 0$ , where we have used the hypothesis  $b^*(m) > 0$ . Therefore,  $B(m)$  is strictly increasing if  $b^*(m) > 0$ .

(2) The optimal choice  $b^*$  satisfies the first-order condition (3.8). If  $\phi(m) > 0$ , then  $V'(\phi(m))$  exists, and the optimal choice  $x^*$  satisfies the first-order condition (3.9).

For any given  $(x, m)$ , the objective function in a buyer's problem (3.6) is differentiable with respect to  $b$ . Thus, if the optimal choice  $b^*$  is interior, it satisfies the first-order condition (3.8). Now consider the optimal choice  $x^*$  and assume  $\phi(m) > 0$  (i.e.,  $x^*(m) < m$ ). Since  $x^*(m) < m$ , a procedure similar to the derivation of  $J'(a)$  in part (1) above but applied to (3.6) yields:

$$\begin{aligned} B'(m^+) &= \beta [b^*(m)V'(\phi^+(m)) + (1 - b^*(m))V'(m^+)] \\ B'(m^-) &= \beta [b^*(m)V'(\phi^-(m)) + (1 - b^*(m))V'(m^-)], \end{aligned}$$

where  $\phi^+(m) = m^+ - x^*(m)$  and  $\phi^-(m) = m^- - x^*(m)$ . Comparing these equations with (C.1) and (C.2) yields that  $V'(\phi^+(m)) = V'(\phi^-(m)) = V'(\phi)$  which is given by (3.9).

(3) For any  $m \in [k, \bar{m}]$  such that  $b^*(m) > 0$ , if  $B(m) = V(m)$  and if there exists a neighborhood  $O \ni m$  such that  $B(m') \leq V(m')$  for all  $m' \in O$ , then  $B'(m)$  and  $V'(m)$  exist and satisfy (3.10) in part (iv) of Theorem 3.2. Also, (3.10) holds for  $m = \bar{m}$  if  $B'(\bar{m}) = V'(\bar{m})$ .

Take any  $m \in [k, \bar{m})$  that satisfies the hypotheses described in this part. Because  $B(m') \leq V(m')$  for all  $m' \in O(m)$  and  $B(m) = V(m)$ , continuity of  $B$  and  $V$  implies that  $B'(m^-) \geq V'(m^-)$  and  $B'(m^+) \leq V'(m^+)$ . Substituting  $V'(m^-) \leq B'(m^-)$  into the right-hand side of (C.2), we get:

$$V'(m^-) \leq B'(m^-) \leq \frac{b^*(m)}{1 - \beta[1 - b^*(m)]} u_1(x^*(m), b^*(m)). \quad (\text{C.5})$$

Similarly, substituting  $V'(m^+) \geq B'(m^+)$  into the right-hand side of (C.1), we get:

$$V'(m^+) \geq B'(m^+) \geq \frac{b^*(m)}{1 - \beta[1 - b^*(m)]} u_1(x^*(m), b^*(m)). \quad (\text{C.6})$$

On the other hand, concavity of  $V$  implies  $V'(m^-) \geq V'(m^+)$ . Thus,  $V'(m^-) = B'(m^-) = B'(m^+) = V'(m^+)$ . Moreover,  $V'(m)$  and  $B'(m)$  satisfy (3.10).

If  $m = \bar{m}$ , it is still true that  $B'(m^-) \geq V'(m^-)$ , and so (C.5) also holds at  $m = \bar{m}$ . However, since we cannot presume  $V'(\bar{m}^+) \geq B'(\bar{m}^+)$ , we cannot conclude that (C.6) holds at this point. If  $B'(\bar{m}) = V'(\bar{m})$ , however, (C.1) and (C.2) imply that (3.10) holds at  $m = \bar{m}$ .

(4) Consider any  $m \in [k, \bar{m})$  such that  $b^*(m) > 0$  and  $\phi(m) > 0$ . If  $B(m) = V(m)$  and if there exists a neighborhood  $O$  surrounding  $m$  such that  $B(m') \leq V(m')$  for all  $m' \in O$ , then  $b^*$  and  $\phi$  are strictly increasing at  $m$  and  $V$  is strictly concave at  $\phi(m)$ , with  $V(\phi(m)) = B(\phi(m))$  and  $V'(\phi(m)) > V'(m)$ . These properties also hold for  $m = \bar{m}$  if  $B'(\bar{m}) = V'(\bar{m})$ .

Take any arbitrary  $m_1 \in [k, \bar{m}]$  that satisfies the hypotheses for  $m$  described in this paper. If  $m_1 = \bar{m}$ , then add the assumption  $B'(m_1) = V'(m_1)$ . Shorten the notation  $\phi(m_1)$  as  $\phi_1$  and  $b^*(m_1)$  as  $b_1^*$ . As a preliminary step, we prove  $V'(\phi_1) > V'(m_1)$  so that  $V$  must be strictly concave in some sections of  $[\phi_1, m_1]$ . By the construction of  $m_1$ ,  $V'(m_1)$  satisfies (3.10) and  $V'(\phi_1)$  satisfies (3.9). Subtracting the two relations yields:

$$V'(\phi_1) - V'(m_1) = \frac{1 - \beta}{\beta[1 - \beta(1 - b_1^*)]} u_1(x_1^*, b_1^*) > 0.$$

Next, we prove that  $b^*(\cdot)$  is strictly increasing at  $m_1$ . Let  $m_2$  be sufficiently close to  $m_1$  so that  $\phi(m_2) > 0$  and  $b^*(m_2) > 0$  (which is feasible because  $\phi(m)$  and  $b(m)$  are continuous functions). Shorten the notation  $(x^*(m_i), b^*(m_i), \phi(m_i))$  to  $(x_i^*, b_i^*, \phi_i)$ , where  $x_i^* = m_i - \phi_i$  and  $i = 1, 2$ . Since the proofs of strict monotonicity of  $b^*(m_1)$  at  $m_1$  are similar in the cases  $m_2 > m_1$  and  $m_2 < m_1$ , let us consider only the case  $m_2 > m_1$ . By Lemma 3.3,  $x_2^* \geq x_1^*$ ,  $b_2^* \geq b_1^*$  and  $\phi_2 \geq \phi_1$ . We prove that  $b_2^* > b_1^*$ . Because  $b_i^* > 0$ , the first-order condition for  $b$ , (3.8), yields:

$$u(x_i^*, b_i^*) + \beta[V(\phi_i) - V(m_i)] + b_i^* u_2(x_i^*, b_i^*) = 0.$$

Subtracting these conditions for  $i = 1, 2$ , and re-organizing, we get:

$$\begin{aligned} & u(x_1^*, b_1^*) - u(x_1^*, b_2^*) - b_2^* u_2(x_2^*, b_2^*) + b_1^* u_2(x_1^*, b_1^*) \\ = & \{u(x_2^*, b_2^*) + \beta[V(m_2 - x_2^*) - V(m_2)] - u(x_1^*, b_2^*) - \beta[V(m_2 - x_1^*) - V(m_2)]\} \\ & + \beta\{V(m_1) - V(m_1 - x_1^*) + V(m_2 - x_1^*) - V(m_2)\}. \end{aligned}$$

The term in the first pair of braces on the right-hand side is equal to  $[R(x_2^*, b_2^*, m_2) - R(x_1^*, b_2^*, m_2)]$ , where  $R$  is defined by (3.13). This term is non-negative because  $x_2^*$  maximizes  $R(x, b_2^*, m_2)$  and  $x_1^*$  is a feasible choice of  $x$  in this maximization problem (as  $x_1^* \leq x_2^*$ ). The term in the second pair of braces is strictly positive because  $V$  is strictly concave in some sections of  $[\phi_1, m_1] \subset [\phi_1, m_2]$ , as proven earlier. Thus, the left-hand side of the above equation must be strictly positive. Noting that  $u_2(x_2^*, b_2^*) \geq u_2(x_1^*, b_2^*)$  (because  $u_{12} > 0$  and  $x_2^* \geq x_1^*$ ), we get:

$$u(x_1^*, b_1^*) - u(x_1^*, b_2^*) - b_2^* u_2(x_2^*, b_2^*) + b_1^* u_2(x_1^*, b_1^*) > 0.$$

The left-hand side of this inequality is a strictly increasing function of  $b_2^*$ , and it is equal to 0 when  $b_2^* = b_1^*$ . Thus, the inequality implies  $b_2^* > b_1^*$ .

Let us complete the proof of part (4). Since (3.10) and (3.9) hold at  $m = m_1$ , we can combine the two relations to obtain:

$$V'(\phi_1) = V'(m_1) \left[ \frac{1 - \beta}{\beta b^*(m_1)} + 1 \right].$$

Because  $b^*(m_1)$  is strictly increasing at  $m_1$  and  $V$  is concave, the right-hand side above is strictly decreasing in  $m_1$ . Thus, the above equation requires  $V$  to be strictly concave at  $\phi(m_1)$  and  $\phi$  to be strictly increasing at  $m_1$ . **QED**

## D. Proof of Theorem 3.5

We prove each part of Theorem 3.5 in turn.

**Part (i).** Let us express the functional equation (2.1) in a worker's problem as  $W(m) = T_W V(m)$  for  $m \in [0, \bar{m}]$ , and express the functional equation (2.2) in a buyer's problem as  $B(m) = T_B V(m)$  for  $m \in [0, \bar{m}]$ . Substituting these expressions into (2.3) to obtain  $\tilde{V}$ , we can rewrite (2.4) as  $V(m) = TV(m)$ , where

$$TV(m) \equiv \max_{(z_1, z_2, \pi_1, \pi_2)} [\pi_1 \max\{T_W V(z_1), T_B V(z_1)\} + \pi_2 \max\{T_W V(z_2), T_B V(z_2)\}] \quad (\text{D.1})$$

$$\begin{aligned} \text{s.t.} \quad & \pi_1 z_1 + \pi_2 z_2 = m, \quad \pi_1 + \pi_2 = 1, \quad z_2 \geq z_1, \\ & \pi_j \in [0, 1] \text{ and } z_j \geq 0 \text{ for } j = 1, 2. \end{aligned}$$

Lemma 3.1 proves that  $T_W$  maps  $\mathcal{V}[0, \bar{m}]$  into  $\mathcal{V}[0, \bar{m}]$ ; i.e.,  $T_W$  maps the set of continuous, increasing and concave functions on  $[0, \bar{m}]$  into itself. Theorem 3.2 proves that  $T_B$  maps  $\mathcal{V}[0, \bar{m}]$



into  $\mathcal{C}[0, \bar{m}]$  (but not necessarily into  $\mathcal{V}[0, \bar{m}]$ ). Thus, the objective function in (D.1) is a continuous function of  $z_1$  and  $z_2$ . Also, the objective function is increasing in  $\pi_1$  and  $\pi_2$ , and the feasibility set in the above problem is increasing in  $m$ . These features of the maximization problem above imply that  $T$  maps  $\mathcal{V}[0, \bar{m}]$  into  $\mathcal{C}[0, \bar{m}]$ . Moreover, since the function  $\max\{T_W V(z), T_B V(z)\}$  is continuous in  $z$  on a closed interval  $[0, \bar{m}] \ni z$ , the lottery in (2.4) makes  $TV(m)$  a concave function (see Appendix F in Menzio and Shi, 2010, for a proof). Thus,  $T$  is a self-map on  $\mathcal{V}[0, \bar{m}]$ .

It is evident from (2.1) and (2.2) that  $T_W$  and  $T_B$  are monotone mappings, and so  $T$  is a monotone mapping. It is also easy to verify that  $T_W$  and  $T_B$  feature discounting with a factor  $\beta \in (0, 1)$ , and so does  $T$ . Hence,  $T$  satisfies Blackwell's sufficient conditions for a monotone contraction mapping. Moreover, because  $\mathcal{V}[0, \bar{m}]$  is a closed subset of the complete metric space  $\mathcal{C}[0, \bar{m}]$ ,  $T$  has a unique fixed point  $V \in \mathcal{V}[0, \bar{m}]$  (see Stokey et al., 1989).

**Part (ii).** For a worker with any balance  $m$ , the choice of working zero hours yields the value  $\beta V(m)$ . Because this choice is always feasible,  $W(m) \geq \beta V(m)$  for all  $m$ . For a buyer who holds  $m \leq k$ , the value is  $B(m) = \beta V(m) \leq W(m)$ . It is clear that  $V(0) = \tilde{V}(0) = W(0)$ . Also,  $V(0) \geq 0$ , because an individual without money can always choose not to trade. To prove  $V(0) > 0$ , suppose  $V(0) = 0$ , to the contrary. In this case,  $0 = V(0) \geq W(0) \geq \beta V(0) = 0$ , and so  $W(0) = V(0) = 0$ . Using the definition of  $W(0)$ , we have  $\beta V(\hat{m}) - h(\hat{m}) = 0$ . Since this equation has a unique solution and since  $\hat{m} = 0$  satisfies the equation, then  $\hat{m} = 0$ . Recall that  $\hat{m} = \ell^*(0)$  is the optimal labor supply of an individual without money and that the policy function  $\ell^*(m)$  is decreasing in  $m$ . Thus,  $\hat{m} = 0$  implies that  $\ell^*(m) = 0$  for all  $m \geq 0$ . In this case, no individual will work for money, and so a monetary equilibrium does not exist. Therefore, for a monetary equilibrium to exist, it must be the case that  $V(m) \geq W(m) \geq W(0) = V(0) > 0$  for all  $m$ .

We now prove that  $V(m) > W(m)$  for all  $m > 0$ . For all  $m > 0$  such that the constraint  $y^* \geq m$  is binding for a worker, (3.2) yields  $W(m) = \beta V(m) < V(m)$ . Now consider  $m > 0$  such that the constraint  $y^* \geq m$  is not binding for a worker. Contrary to the result in this part, suppose  $V(\tilde{m}) = W(\tilde{m})$  for some  $\tilde{m} > 0$  such that  $y^*(\tilde{m}) > \tilde{m}$ . Since  $\beta V'(y^*(\tilde{m})) = W'(\tilde{m})$  by (3.1) in Lemma 3.1, then  $V'(y^*(\tilde{m})) > 0$ , and concavity of  $V$  implies  $V'(\tilde{m}^-) > 0$ . In this case,

$$V'(\tilde{m}^-) \leq W'(\tilde{m}) = \beta V'(y^*(\tilde{m})) \leq \beta V'(\tilde{m}^-) < V'(\tilde{m}^-).$$

The first inequality follows from the hypothesis  $V(\tilde{m}) = W(\tilde{m})$  and the fact  $V(m) \geq W(m)$  for all  $m < \tilde{m}$ . The equality is from (3.1). The second inequality follows from concavity of  $V$ , and the last inequality from  $V'(\tilde{m}^-) > 0$ . Since the above result is a contradiction, we conclude that  $V(m) > W(m)$  for all  $m > 0$ .

**Part (iii).** We prove first that there is some  $m' \in (0, \infty)$  such that  $B(m') > W(m')$ . Suppose,

to the contrary, that  $B(m) \leq W(m)$  for all  $m \in (0, \infty)$ . Then,  $\tilde{V}(m) = W(m)$  for all  $m$ . Since  $W(m)$  is concave (see Lemma 3.1),  $\tilde{V}(\cdot)$  is concave in this case, and so  $V(m) = \tilde{V}(m) = W(m)$  for all  $m$ . In this case, (3.2) yields

$$V(m) = \max_{y \geq m} [\beta V(y) - h(y - m)], \quad \text{all } m > 0.$$

If  $y^*(m) = m$ , the above equation yields  $V(m) = 0$ , which contradicts part (ii) above. If  $y^*(m) > m$ , (3.1) in Lemma 3.1 implies that  $W$  is differentiable at  $m$ , with  $W'(m) = \beta V'(y^*(m)) > 0$ . Since  $W(m) = V(m)$  for all  $m > 0$  in this case,  $V'(m) = W'(m) = \beta V'(y^*(m)) \leq \beta V'(m)$ . This implies  $V'(m) = 0 = V'(y^*(m))$ , which contradicts  $V'(y^*(m)) > 0$ .

Next, we prove that there exists  $m_0 \in (k, \bar{m}]$  with  $V(m_0) = B(m_0)$  such that an individual with  $m < m_0$  will play the lottery with the prize  $m_0$ . For an individual with a balance  $m \in (0, k)$ , the lottery with  $z_1 = 0$  and  $z_2 = m'$  yields a value higher than  $\tilde{V}(m)$ , where  $m'$  is described above. Thus, these individuals will participate in lotteries. However,  $m'$  may not necessarily be the optimal prize of the lottery for these individuals. The optimal prize is  $m_0$ , defined by (3.16). Clearly,  $m_0 > k > 0$ ,  $V(m_0) = \tilde{V}(m_0) = B(m_0)$ , and  $V(m) \geq \tilde{V}(m)$  for all  $m \in [0, m_0]$ .

Now we prove that  $b^*(m_0) > 0$  and  $\phi(m_0) = 0$ . Suppose  $b^*(m_0) = 0$  to the contrary, and so  $B(m_0) = \beta V(m_0)$ . Since  $V(m_0) = B(m_0)$ , as shown above, then  $V(m_0) = 0$ , which contradicts the above result in part (ii) that  $V(m) > 0$  for all  $m \geq 0$ . Thus, it must be true that  $b^*(m_0) > 0$ . Since  $V(m_0) = B(m_0)$ , (C.5) holds for  $m = m_0$  which, together with  $b^*(m_0) > 0$ , implies  $V'(m_0^-) < u_1(x^*(m_0), b^*(m_0)) / \beta$ . Since  $V(m)$  is linear for  $m \in [0, m_0]$ , then  $V'(\phi(m_0)) = V'(m_0^-) < u_1(x^*(m_0), b^*(m_0)) / \beta$ . If  $\phi(m_0) > 0$ , then (3.9) holds for  $m = m_0$ , which yields the contradiction that  $V'(\phi(m_0)) = u_1(x^*(m_0), b^*(m_0)) / \beta$ . Thus, it must be true that  $\phi(m_0) = 0$ .

Finally, since  $V(m_0) = B(m_0)$  and  $b^*(m_0) > 0$ ,  $m_0$  satisfies the hypotheses in part (iv) of Theorem 3.2 if  $m_0 < \bar{m}$ . Thus, if  $m_0 < \bar{m}$ , then (3.10) holds for  $m = m_0$ , which implies  $V'(m_0) = B'(m_0) > 0$ .

**Part (iv).** We first prove that  $V'(m)$  exists for all  $m \in [0, \bar{m})$  and  $B'(m)$  exists for all  $m \in [k, \bar{m})$  such that  $b^*(m) > 0$ . If  $V'(m)$  exists for all  $m \in [0, \bar{m})$ , then part (iii) of Theorem 3.2 implies that  $B'(m)$  exists for all  $m \in [k, \bar{m})$  such that  $b^*(m) > 0$ . To prove that  $V'(m)$  exists for all  $m \in [0, \bar{m})$ , note that the lottery with the prize  $m_0$  implies that  $V'(m)$  exists for all  $m \in [0, m_0)$ . If  $m_0 = \bar{m}$ , then  $V'(m)$  exists for all  $m \in [0, \bar{m})$ . If  $m_0 < \bar{m}$ , then  $V'(m_0)$  also exists, as shown in part (iii) above. What remains to be proven is that  $V'(m)$  exists for all  $m \in (m_0, \bar{m})$ . Suppose to the contrary that  $V'(\tilde{m})$  does not exist for some  $\tilde{m} \in (m_0, \bar{m})$ . In this case,  $V'(\tilde{m}^-) > V'(\tilde{m}^+)$ , and so  $V$  is strictly concave at  $\tilde{m}$ . Because  $V(m) > W(m)$  for all  $m > 0$ , as proven in part (ii) above, we must have  $V(\tilde{m}) = B(\tilde{m})$ . Also,  $b^*(\tilde{m}) \geq b^*(m_0) > 0$ . Thus, the hypotheses in part

(iv) of Theorem 3.2 are true for  $m = \tilde{m}$ , and so  $V'(\tilde{m})$  exists. This contradicts the supposition that  $V'(\tilde{m})$  does not exist.

Next, we prove that  $V'(m) > 0$  for all  $m \in [0, \bar{m}]$ . For all  $m \in [0, m_0)$ ,  $V(m)$  is linear and  $V'(m) = V'(m_0^-) > 0$ . If  $m_0 = \bar{m}$ , then  $V'(m) > 0$  for all  $m \in [0, \bar{m}]$ . If  $m_0 < \bar{m}$ , then  $V'(m_0) = B'(m_0) > 0$ , as proven in part (iii) above. We need to prove  $V'(m) > 0$  for all  $m \in [m_0, \bar{m}]$ . Consider any  $m > m_0$ . Since  $b^*(m_0) > 0$  and  $b^*(m)$  is an increasing function (see part (i) of Theorem 3.2), then  $b^*(m) > 0$ , which further implies that  $B(m)$  is strictly increasing (see part (iii) of Theorem 3.2). Because  $\tilde{V}(m) = B(m)$  for all  $m \geq m_0$ , then  $\tilde{V}(m)$  is strictly increasing over such  $m$ . Recall that  $V(m)$  is constructed with lotteries over  $\tilde{V}(m)$ . If  $V(m_1) = V(m_2)$  for some  $m_2 > m_1 > m_0$ , contrary to the claimed result, then  $V(m)$  must be constant for all  $m \in [m_1, m_2]$ . Extend this interval to  $[m'_1, m'_2]$ , with  $m'_1 \leq m_1$  and  $m'_2 \geq m_2$ , so that  $V(m'_1) = \tilde{V}(m'_1)$  and  $V(m'_2) = \tilde{V}(m'_2)$ . Then,  $\tilde{V}(m'_2) = V(m_2) = V(m_1) = \tilde{V}(m'_1)$ , which contradicts strict monotonicity of  $\tilde{V}$ .

**Part (v).** For each exogenous upper bound on individuals' real balances, the policy and value functions are characterized by the results in section 3 up to part (iv) of the current theorem. Now we vary the upper bound, possibly to  $\infty$ , and prove that individuals' real balances in the equilibrium are indeed bounded above by a finite  $\bar{m}$  that satisfies the current part of the theorem. Note first that the balance obtained by a worker is  $\hat{m} = \ell^*(0) \leq 1$ , which is clearly bounded above. If  $B(\hat{m}) = V(\hat{m})$  and  $B'(\hat{m}) = V'(\hat{m})$ , then  $z_2^*(\hat{m}) = \hat{z}_2 = \hat{m}$ , in which case we can set  $\bar{m} = \hat{m}$  as the upper bound to satisfy the properties stated in the current part of the theorem. In the remainder of this proof, suppose  $B(\hat{m}) < V(\hat{m})$ , and so a lottery is played at  $\hat{m}$ . Set the upper bound  $\bar{m}$  in the analysis up to part (iv) of the theorem to an arbitrary finite number  $\bar{m} > \hat{m}$ . Given this arbitrary bound  $\bar{m}$ , it may or may not be true that  $z_2^*(\hat{m}) < \bar{m}$ . By varying the arbitrary bound, we can re-define  $\bar{m}$  as the least upper bound above which  $z_2^*(\hat{m}) < \bar{m}$ . If this least upper bound is finite, then it satisfies the properties stated in the current part of the theorem. If this least upper bound is infinite, then  $z_2^*(\hat{m}) = \bar{m}$  for all  $\bar{m} > \hat{m}$ , in which case the lottery at  $\hat{m}$  is not well-defined for endogenously determined  $\bar{m}$ . It suffices to show that this unbounded case does not arise in the equilibrium. The unbounded case occurs only if there exists a finite  $m_1 > \hat{m}$  such that the following two conditions are satisfied:

- (A)  $B(m)$  is strictly increasing and (weakly) convex for all  $m \geq m_1$ ;
- (B) for every  $m_2 \geq m_1$ , there exists  $z_1 < \hat{m}$  such that for all  $m < m_2$ ,  $B(m)$  lies below or on the extension of the line connecting  $B(z_1)$  and  $B(m_2)$ .

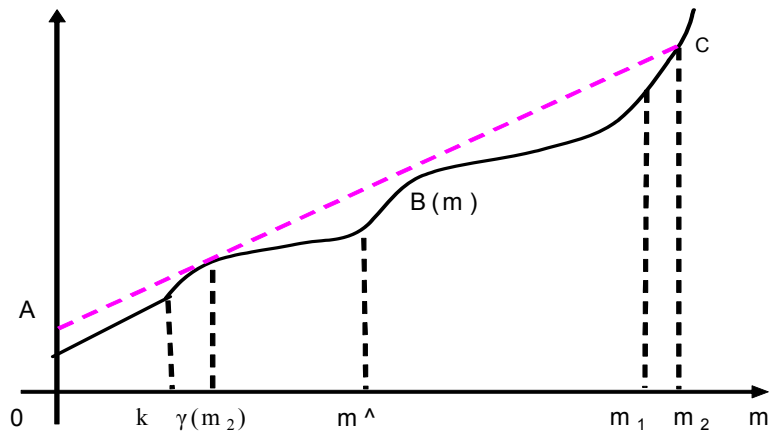


Figure 3.1

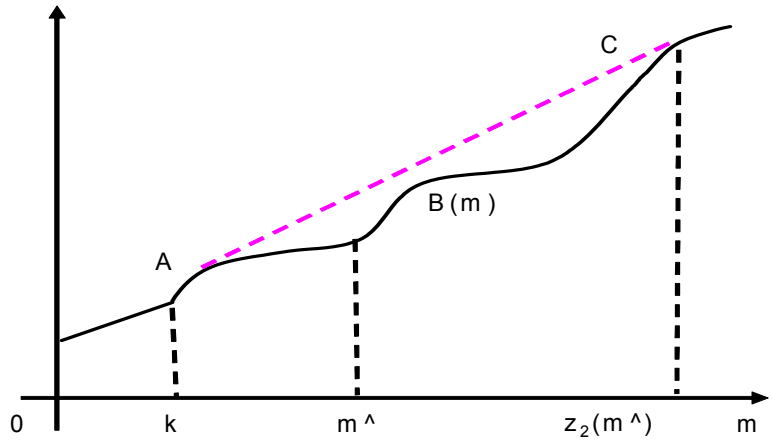


Figure 3.2

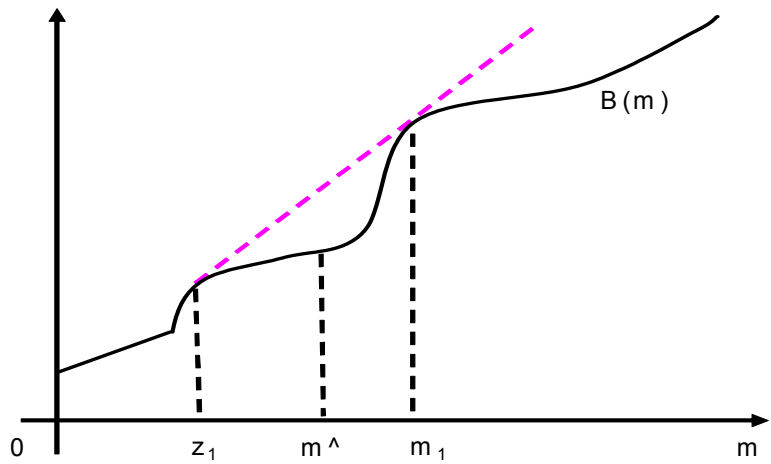


Figure 3.3

Figure 3.1 depicts this unbounded case. If (A) is violated, as depicted in Figure 3.2, then there must exist a finite number  $m_1 > \hat{m}$  such that  $B(m)$  is concave for all  $m \geq m_1$ . In this case, the high prize of the lottery at  $\hat{m}$  is  $z_2^*(\hat{m}) < \infty$ , and so we can set  $\bar{m} = z_2^*(\hat{m})$  as the upper bound stated in the current part of Theorem 3.5. If (B) is violated, as depicted in Figure 3.3, then there must exist a finite  $m_1 > \hat{m}$  and an associated  $z_1 < \hat{m}$  such that the low prize of the lottery at  $\hat{m}$  is  $z_1$  and the high prize is  $m_1$  and that, for all  $m > m_1$ , the function  $B(m)$  lies below or on the extension of the line connecting  $B(z_1)$  and  $B(m_1)$ . In this case,  $\bar{m} = m_1$  satisfies the properties in this part of Theorem 3.5. Note that in the case depicted in Figure 3.3,  $B(m)$  can still be strictly increasing and convex for sufficiently large  $m$ , but such a section of  $B$  is irrelevant in the equilibrium because an individual's balance never goes above  $m_1$ . Also note that the requirement  $z_1 < \hat{m}$  in (B) is important, since the case depicted in Figure 3.3 would not violate (B) if this requirement were not imposed.

Suppose, to the contrary, that there exists a finite  $m_1 > \hat{m}$  such that (A) and (B) above are satisfied, as depicted in Figure 3.1. We prove that this leads to the contradiction that  $B(m)$  is uniformly bounded. Consider any arbitrary  $m_2 \geq m_1$ . When individuals' real balances are exogenously capped by  $m_2$ , the lottery at  $\hat{m}$  is well-defined, with  $m_2$  as the high prize, and all characterizations of the policy and value functions that we have obtained so far (including parts (i)-(iv) of the current Theorem 3.5) remain valid with  $\bar{m} = m_2$ . However, since  $B'(m_2) > V'(m_2)$  in this case, we have  $B'(\bar{m}) > V'(\bar{m})$ . Denote the low prize of the lottery at  $\hat{m}$  as  $\zeta(m_2) = z_1^*(\hat{m})$  so as to emphasize its dependence on the exogenous upper bound  $m_2$ . Without loss of generality, assume that  $\hat{m} \geq m_0$ , i.e.,  $B(\zeta(m_2)) = V(\zeta(m_2))$ . (If  $B(\zeta(m_2)) < V(\zeta(m_2))$ , then  $\zeta(m_2) = 0$ , in which case the proof is still valid after replacing  $V(\zeta(m_2))$  below with  $V(0)$ .) Denote

$$\alpha(m_2) = \frac{B(m_2) - V(\zeta(m_2))}{m_2 - \zeta(m_2)},$$

$$\hat{V}(m, m_2) = B(m_2) - \alpha(m_2)(m_2 - m), \quad m \in [0, m_2].$$

Here,  $\alpha(m_2)$  is the slope of the line connecting  $B(\zeta(m_2))$  and  $B(m_2)$ , and  $\hat{V}$  is the extension of this line to the domain  $[0, m_2]$  (the dashed line from point A to point C in Figure 3.1).

We prove that  $\alpha(m_2)$  is increasing for all  $m_2 \geq m_1$ . Take any arbitrary  $m' > m_2 \geq m_1$ . By definition,  $\zeta(m') < \hat{m} \leq m_2$ . So,  $B(\zeta(m'))$  lies below or on the line  $\hat{V}(m, m_2)$ ; i.e.,  $\hat{V}(\zeta(m'), m_2) \geq B(\zeta(m')) = V(\zeta(m'))$ . Also, since  $B(m)$  is increasing and convex for all  $m \geq m_1$ , and  $m' > m_2$ , then  $B(m') \geq \hat{V}(m', m_2)$ . Using these two results, we have:

$$\alpha(m') \geq \frac{\hat{V}(m', m_2) - V(\zeta(m'))}{m' - \zeta(m')} \geq \frac{\hat{V}(m', m_2) - \hat{V}(\zeta(m'), m_2)}{m' - \zeta(m')} = \alpha(m_2),$$

where the last equality comes from substituting the expression for  $\hat{V}$ . Thus,  $\alpha(\cdot)$  is increasing.

Now we derive a uniform upper bound on  $B(m_2)$  for all  $m_2 \geq m_1$ . It is clear that  $V(m) \leq \hat{V}(m, m_2)$  for  $m \in [0, m_2]$ , with equality if  $m \in [\zeta(m_2), m_2]$ . We have:

$$\begin{aligned}
B(m_2) &= \max_{b \in [0,1], x \in [0, m_2]} \{b[u(x, b) + \beta V(m_2 - x)] + (1 - b)\beta V(m_2)\} \\
&\leq \max_{b \in [0,1], x \in [0, m_2]} \left\{ b \left[ u(x, b) + \beta \hat{V}(m_2 - x, m_2) \right] + (1 - b)\beta \hat{V}(m_2, m_2) \right\} \\
&= \beta B(m_2) + \max_{b \in [0,1], x \in [0, m_2]} b[u(x, b) - \beta \alpha(m_2)x] . \\
&\leq \beta B(m_2) + \max_{b \in [0,1], q \geq 0} b \left\{ U(q) - \beta \alpha(m_2) \left[ q + \frac{k}{\mu(b)} \right] \right\} .
\end{aligned}$$

The first inequality follows from  $V(m) \leq \hat{V}(m, m_2)$  for all  $m \in [0, m_2]$ . The ensuing equality comes from the linearity of  $\hat{V}$  and  $\hat{V}(m_2, m_2) = B(m_2)$ . The last inequality comes from the fact that if we ignore the constraint  $x \leq m_2$  in the maximization problem, the resulted maximum cannot be smaller. Here, we have substituted the relationship  $x = q + \frac{k}{\mu(b)}$  and used  $(q, b)$  as the choices. Similarly, because  $b \leq 1$  and  $\mu(b) \leq 1$ , the resulted maximum cannot be smaller if we set  $b = 1$  and  $\mu(b) = 1$ . Thus,

$$B(m_2) \leq \frac{D(\alpha(m_2))}{1 - \beta} \quad \text{where } D(\alpha(m_2)) \equiv \max_{q \geq 0} [U(q) - \beta \alpha(m_2)(q + k)] . \quad (\text{D.2})$$

The notation  $D(\alpha(m_2))$  emphasizes the fact that  $D$  depends on  $m_2$  only through  $\alpha(m_2)$ . Because  $U'(q)$  is strictly decreasing and  $U'(\infty) = 0$ , the solution for  $q$  to the maximization problem in (D.2) is unique, strictly positive and finite for all  $\alpha < \infty$ . So,  $D(\alpha) < \infty$  for all  $\alpha < \infty$ . Applying the envelope condition, we have  $D'(\alpha) < 0$ . Because  $\alpha(\cdot)$  is increasing, as shown above, then  $\alpha(m_2) \geq \alpha(m_1) > 0$  and  $D(\alpha(m_2)) \leq D(\alpha(m_1)) < \infty$ . Therefore,  $B(m_2) \leq D(\alpha(m_1))/(1 - \beta) < \infty$  for all  $m_2 \geq m_1$ . This result contradicts the supposition that  $B(m)$  is strictly increasing and convex for all  $m \geq m_1$ . **QED**

## E. Proof of Lemma 4.1

Part (i) of the lemma is implied by part (iii) of Theorem 3.5, with  $m = \hat{m}$ . Part (ii) of the lemma is obvious if  $\hat{m} < m_0$  and, if  $\hat{m} \geq m_0$ , it is implied by part (iii) of the lemma. In particular, since part (iii) implies that  $B(\phi^i(\hat{z}_j)) = V(\phi^j(\hat{z}_j))$ ,  $B'(\phi^i(\hat{z}_j)) = V'(\phi^j(\hat{z}_j))$  and  $V$  is strictly concave at  $\phi^i(\hat{z}_j)$  for all  $i$  in the set  $\{0, 1, 2, \dots, \hat{n}_j - 1\}$ , then  $\phi^i(\hat{z}_j) \geq m_0$  and a buyer with the balance  $\phi^i(\hat{z}_j)$  has no need for a lottery for any  $i$  in the aforementioned set. Thus, the only lottery possibly played in the steady state is the one at  $\hat{m}$ .

We use induction to prove parts (a) and (b) of part (iii) of the lemma. Assume  $\hat{m} \geq m_0$ , as is required in part (iii), and take  $\hat{z}_j$  as either prize of the lottery at  $\hat{m}$ . Start with  $i = 1$ . Because  $\hat{m} \geq m_0$ , then  $\hat{z}_j \geq m_0$ , and so  $b^*(\hat{z}_j) \geq b^*(m_0) > 0$ , where the strict inequality comes from

part (iii) of Theorem 3.5. Thus, part (a) holds true for  $i = 1$ . Moreover, by the construction of the lottery at  $\hat{m}$ ,  $B(\hat{z}_j) = V(\hat{z}_j)$  and  $B'(\hat{z}_1) = V'(\hat{z}_1)$ . Also, since  $\hat{z}_2 = \bar{m}$ , part (v) of Theorem 3.5 implies  $B'(\hat{z}_2) = V'(\hat{z}_2)$ . Thus,  $m = \hat{z}_j$  satisfies the hypotheses in part (v) of Theorem 3.2 which implies that, if  $\phi(\hat{z}_j) > 0$ , then  $V$  is strictly concave at  $\phi(\hat{z}_j)$ . Strict concavity of  $V$  at  $\phi(\hat{z}_j)$  implies  $V(\phi(\hat{z}_j)) = B(\phi(\hat{z}_j))$ : if  $B(\phi(\hat{z}_j)) < V(\phi(\hat{z}_j))$ ,  $V$  around  $\phi(\hat{z}_j)$  would be a linear segment generated by the lottery in (2.4), which would contradict strict concavity of  $V$  at  $\phi(\hat{z}_j)$ . Thus,  $m = \phi(\hat{z}_j)$  satisfies the hypotheses in part (iv) of Theorem 3.2 which implies  $V'(\phi(\hat{z}_j)) = B'(\phi(\hat{z}_j))$ . Moreover, strict concavity of  $V$  at  $\phi(\hat{z}_j)$  implies that  $\phi(\hat{z}_j) \geq m_0$ , because  $V$  is linear for all  $m < m_0$ . Thus, part (b) holds true for  $i = 1$  if  $\phi(\hat{z}_j) > 0$ . If  $\phi(\hat{z}_j) = 0$ , on the other hand, part (b) is vacuous.

Suppose that parts (a) and (b) hold for an arbitrary  $i \in \{1, 2, \dots, \hat{n}_j - 1\}$ , we prove that they hold for  $i + 1$  and, by induction, they hold for all  $i \in \{1, 2, \dots, \hat{n}_j - 1\}$ . Because  $\phi^i(\hat{z}_j) \geq m_0$  by the supposition,  $b^*(\phi^i(\hat{z}_j)) \geq b^*(m_0) > 0$ , and so part (a) holds for  $i + 1$ . If  $i = \hat{n}_j - 1$ , then part (b) is vacuous for  $i + 1$ . If  $i < \hat{n}_j - 1$ , then  $\phi^{i+1}(\hat{z}_j) > 0$ . Since  $V(\phi^i(\hat{z}_j)) = B(\phi^i(\hat{z}_j))$  and  $V$  is strictly concave at  $\phi^i(\hat{z}_j)$ , by the supposition, then  $m = \phi^i(\hat{z}_j)$  satisfies the hypotheses in part (v) of Theorem 3.2 which implies that  $V$  is strictly concave at  $\phi^{i+1}(\hat{z}_j)$ . Strict concavity of  $V$  at  $\phi^{i+1}(\hat{z}_j)$  implies  $V(\phi^{i+1}(\hat{z}_j)) = B(\phi^{i+1}(\hat{z}_j))$  and  $\phi^{i+1}(\hat{z}_j) \geq m_0$ . Thus,  $m = \phi^{i+1}(\hat{z}_j)$  satisfies the hypotheses in part (iv) of Theorem 3.2 which implies  $V(\phi^{i+1}(\hat{z}_j)) = B(\phi^{i+1}(\hat{z}_j))$ . Hence, part (b) holds for  $i + 1$ .

If  $i = \hat{n}_j$ , part (a) follows from the same proof as above, and part (b) is vacuous.

Finally, suppose  $\phi^{\hat{n}_j}(\hat{z}_j) > 0$ , contrary to part (c). Because part (b) holds for  $i = \hat{n}_j - 1$ , then  $m = \phi^{\hat{n}_j-1}(\hat{z}_j)$  satisfies all the hypotheses in part (v) of Theorem 3.2 which implies that  $V$  is strictly concave at  $\phi^{\hat{n}_j}(\hat{z}_j)$ . A contradiction. **QED**

## F. Proof of Theorem 4.2

The text preceding Theorem 4.2 has established that there exists a unique, block recursive monetary steady state and that the frequency function  $g$  in this steady state is independent of  $\omega$ . The paragraph immediately following Theorem 4.2 shows that the monetary steady state must be block recursive; that is, there is no monetary steady state other than the block recursive one characterized before Theorem 4.2. These results imply that money is neutral in the steady state. Turn to the result that from either  $\hat{z}_j$  ( $j = 1, 2$ ), the frequency function,  $g(\phi^i(\hat{z}_j))$ , is decreasing in  $\phi^i(\hat{z}_j)$ . To prove this result, note that  $\phi^i(\hat{z}_j) = \phi^{i-1}(\hat{z}_j) - x^*(\phi^{i-1}(\hat{z}_j)) < \phi^{i-1}(\hat{z}_j)$  for all  $1 \leq i \leq \hat{n}_j$  and  $j = 1, 2$ . By part (iii) of Theorem 3.5,  $b^*(m_0) > 0$ . For each  $j \in \{1, 2\}$ , Lemma 4.1 implies that  $\phi^i(\hat{z}_j) \geq m_0$  and  $b^*(\phi^i(\hat{z}_j)) > 0$  for all  $1 \leq i \leq \hat{n}_j - 1$ . Thus, for all  $1 \leq i \leq \hat{n}_j - 1$ ,

$\phi^i(\hat{z}_j)$  satisfies part (v) of Theorem 3.2, which implies that  $b^*(\cdot)$  is strictly increasing at  $\phi^i(\hat{z}_j)$  for each  $i$  and  $j$ . With this feature, (4.4) implies that  $g(\phi^i(\hat{z}_j)) > g(\phi^{i-1}(\hat{z}_j))$  for all  $i = 1, 2, \dots, \hat{n}_j - 1$  and  $j = 1, 2$ .

Next, we prove that there exists  $\beta_0 > 0$  such that if  $\beta \leq \beta_0$ , then  $\hat{m} < m_0$  and  $\phi(\hat{z}_2) = 0$ . Let us shorten the notation  $b^*(m_0)$  to  $b_0$  and  $q^*(m_0)$  to  $q_0$ . Define  $\bar{q}(\beta)$  and  $\underline{q}$  by

$$\frac{U(\underline{q})}{U'(\underline{q})} - \underline{q} = k, \quad U'(\bar{q}(\beta)) = \frac{1}{\beta}h'(\bar{q}(\beta)) + k. \quad (\text{F.1})$$

We then define  $\beta_0$  as

$$\beta_0 = \max_{\beta \in [0,1]} \{\beta : \bar{q}(\beta) \leq \underline{q}\}. \quad (\text{F.2})$$

For any  $\beta \in (0, 1]$ , the assumptions on  $U$  and  $h$  imply that  $\bar{q}(\beta)$  and  $\underline{q}$  are well defined. In particular, the assumptions on  $U$  imply that  $[\frac{U(q)}{U'(q)} - q]$  is a strictly increasing function of  $q$  whose value at  $q = 0$  is 0. Moreover,  $\bar{q}(\beta)$  and  $\underline{q}$  have the following features:

- (a)  $\bar{q}'(\beta) > 0$  and  $\lim_{\beta \rightarrow 0} \bar{q}(\beta) = 0 < \underline{q}$ : These follow from the assumptions on  $U$  and  $h$ .
- (b)  $\underline{q} < q_0$ : To verify this feature, note that  $V'(0) = V'(m_0)$ . Since  $V'(m_0)$  satisfies part (v) of Theorem 3.2 with  $m = m_0$ , we have:

$$V'(0) = \frac{b_0 U'(q_0)}{1 - \beta + \beta b_0}, \quad (\text{F.3})$$

where we have substituted  $u_1(m_0, b_0) = U'(q_0)$ . Also, the lottery in (3.16) implies  $V(m) = V(0) + mV'(0)$  for all  $m \in [0, m_0]$ . Substituting  $V(m_0)$  from this result and  $V'(0)$  from (F.3) into the Bellman equation for  $B(m_0)$  ( $= V(m_0)$ ), we obtain:

$$b_0 [U(q_0) - m_0 U'(q_0)] = (1 - \beta) V(0). \quad (\text{F.4})$$

Here, we have substituted  $u_1(m_0, b_0) = U'(q_0)$  and  $u(m_0, b_0) = U(q_0)$ . Because  $V(0) > 0$  by part (ii) of Theorem 3.5 and  $b_0 > 0$ , (F.4) implies  $U(q_0) > m_0 U'(q_0)$ . Because  $m_0 > q_0 + k$  (as  $\mu(b_0) < 1$ ), this result further implies  $\frac{U(q_0)}{U'(q_0)} - q_0 > k = \frac{U(\underline{q})}{U'(\underline{q})} - \underline{q}$ , which is equivalent to  $\underline{q} < q_0$ .

- (c)  $\bar{q}(\beta) > q^*(\hat{m})$  for all  $\beta \in (0, 1]$ : By the definition of  $\bar{q}(\beta)$  in (F.1),  $\bar{q}(\beta) > q^*(\hat{m})$  if and only if  $U'(q^*(\hat{m})) > \frac{1}{\beta}h'(q^*(\hat{m})) + k$ . The latter relation is verified as follows:

$$U'(q^*(\hat{m})) \geq U'(q^*(\hat{z}_2)) > V'(\hat{z}_2) = V'(\hat{m}) = \frac{1}{\beta}h'(\hat{m}) > \frac{1}{\beta}h'(q^*(\hat{m})) + k.$$

The first inequality comes from the fact  $q^*(\hat{m}) \leq q^*(\hat{z}_2)$ . To obtain the second inequality, we apply (3.10) for  $m = \hat{z}_2$ , which yields  $V'(\hat{z}_2) = \frac{b^*(\hat{z}_2)U'(q^*(\hat{z}_2))}{1 - \beta + \beta b^*(\hat{z}_2)}$ . The first equality above comes from the fact that  $V$  is linear between  $\hat{m}$  and  $\hat{z}_2$ , and the second equality above from the definition of  $\hat{m}$ . The last inequality comes from  $\hat{m} = q^*(\hat{m}) + \frac{k}{\mu(b^*(\hat{m}))}$  and  $\mu(b^*(\hat{m})) < 1$ .



Feature (a) implies that the set  $\{\beta \in [0, 1] : \bar{q}(\beta) \leq \underline{q}\}$  is non-empty and that  $\beta_0 > 0$  is well-defined. Moreover,  $\bar{q}(\beta) \leq \underline{q}$  for all  $\beta \leq \beta_0$ . Using features (b) and (c), we conclude that if  $\beta \leq \beta_0$ , then  $q^*(\hat{m}) < \bar{q}(\beta) \leq \underline{q} < q_0$ . Recall that  $q^*(m)$  is an increasing function. Thus, if  $\beta \leq \beta_0$  then  $\hat{m} < m_0$ , in which case  $\phi(\hat{z}_2) = \phi(m_0) = 0$ .

As a preliminary step toward finding a condition for  $\phi(\hat{z}_2) > 0$ , we consider the limit  $\beta \rightarrow 1$  and characterize the optimal choices in more detail. This exercise is guided by the above result that  $\phi(\hat{z}_2) = 0$  if  $\beta$  is small. Note that although  $\lim_{\beta \rightarrow 1} V(m) = \infty$ , the limit of  $(1 - \beta)V(m)$  is strictly positive and finite for all  $m \in [0, \infty)$ . Also, the limit of  $[V(m) - V(0)]$  is finite for all  $m < \infty$ . We characterize in detail the optimal choices of a buyer with the balance  $m_0$  in the limit  $\beta \rightarrow 1$ . First, taking the limit  $\beta \rightarrow 1$  on (F.3) and (F.4) yields:

$$V'(0) = U'(q_0), \tag{F.5}$$

$$b_0 [U(q_0) - m_0 U'(q_0)] = \lim_{\beta \rightarrow 1} [(1 - \beta) V(0)]. \tag{F.6}$$

Second, since  $u_2 = u_1 k \mu' / \mu^2$ , the first-order condition of  $b_0$  (see (3.8)) yields:

$$\frac{U(q_0)}{U'(q_0)} - m_0 + \frac{k \mu'(b_0) b_0}{[\mu(b_0)]^2} = 0, \tag{F.7}$$

where we have used (F.5) for  $V'(0)$ . Substituting  $b_0 = \mu^{-1}(\frac{k}{m_0 - q_0})$  into (F.7), we can prove that  $q_0 = q^*(m_0)$  is a strictly increasing function of  $m_0$ .

We are now ready to prove that  $\phi(\hat{z}_2) > 0$  in the limit  $\beta \rightarrow 1$  if and only if  $m_0 < \hat{z}_2$ . The “only if” part of this statement is apparent, because  $m_0 \geq \hat{z}_2$  implies  $\phi(\hat{z}_2) = \phi(m_0) = 0$ . To prove the “if” part of the statement, we verify that  $\phi(\hat{z}_2) = 0$  implies  $m_0 \geq \hat{z}_2$  in the limit  $\beta \rightarrow 1$ . Suppose  $\phi(\hat{z}_2) = 0$ . Using part (ii) of Theorem 3.2, we deduce that  $\beta V'(0) \leq U'(q^*(\hat{z}_2))$ . Taking the limit  $\beta \rightarrow 1$  and using (F.5), we write this condition as  $q_0 \geq q^*(\hat{z}_2)$ . Because  $q^*(m)$  is strictly increasing at  $m = m_0$ , then  $m_0 \geq \hat{z}_2$ .

The above procedure leads to the conclusion that when  $\beta$  is sufficiently close to one,  $\phi(\hat{z}_2) > 0$  if and only if  $m_0 < \hat{z}_2$ . To characterize the condition  $m_0 < \hat{z}_2$  explicitly, we suppose that the opposite,  $m_0 \geq \hat{z}_2$ , is true. After solving  $q_0$  from (F.8) as  $q_0(\hat{m})$  and  $b_0$  from (F.9) as  $b_0(\hat{m})$ , we will solve the number  $\hat{m}$  from (F.10) as  $m_e$ . Because the supposition  $m_0 \geq \hat{z}_2$  implies  $\hat{m} \leq m_0$ , the supposition leads to a contradiction if  $\hat{m} = m_e$  satisfies  $\hat{m} > m_0$ , i.e., if (4.6) holds. Therefore, if (4.6) holds, then  $m_0 < \hat{z}_2$  and  $\phi(\hat{z}_2) > 0$  for  $\beta$  sufficiently close to one.

To carry out the procedure described above, we suppose  $m_0 \geq \hat{z}_2$  and consider the limit  $\beta \rightarrow 1$ . Since  $m_0 \geq \hat{m}$  in this case, the lottery for low balances implies  $V'(0) = V'(\hat{m})$ . Because

the definition of  $\hat{m}$  in the limit  $\beta \rightarrow 1$  implies  $V'(\hat{m}) = h'(\hat{m})$ , then  $V'(0) = h'(\hat{m})$ . Substituting this result into (F.5), we solve  $q_0 = q_0(\hat{m})$  where

$$q_0(\hat{m}) \equiv U'^{-1}(h'(\hat{m})). \quad (\text{F.8})$$

Substituting  $m_0 = q_0 + \frac{k}{\mu(b_0)}$  and  $q_0 = q_0(\hat{m})$  into (F.7) yields:

$$\frac{k}{\mu(b_0)} - \frac{k\mu'(b_0)b_0}{[\mu(b_0)]^2} = \frac{U(q_0(\hat{m}))}{h'(\hat{m})} - q_0(\hat{m}). \quad (\text{F.9})$$

Since  $\mu'(b) < 0$  and  $1/\mu(b)$  is strictly convex in  $b$ , the left-hand side of (F.9) is strictly increasing in  $b_0$ . Thus, for any given  $\hat{m}$ , (F.9) solves for a unique  $b_0$ . Denote this solution as  $b_0(\hat{m})$ .

Moreover, since  $\hat{m} \leq m_0$  under the supposition  $m_0 \geq \hat{z}_2$ , the lottery for low balances implies that  $V(\hat{m})$  is linear in  $\hat{m}$  and the slope of the line is  $V'(\hat{m}) = h'(\hat{m})$  in the limit  $\beta \rightarrow 1$ . That is,  $V(\hat{m}) - V(0) = \hat{m}h'(\hat{m})$ . On the other hand, in the limit  $\beta \rightarrow 1$ , a worker's Bellman equation yields  $V(\hat{m}) - V(0) = h(\hat{m}) + \lim_{\beta \rightarrow 1} [(1 - \beta)V(0)]$ . Thus,  $\lim_{\beta \rightarrow 1} [(1 - \beta)V(0)] = \hat{m}h'(\hat{m}) - h(\hat{m})$ . Substituting this result and  $b_0 = b_0(\hat{m})$ , we rewrite (F.6) as

$$-\left. \frac{k\mu'(b_0)(b_0)^2}{[\mu(b_0)]^2} \right|_{b_0=b_0(\hat{m})} = \hat{m} - \frac{h(\hat{m})}{h'(\hat{m})}. \quad (\text{F.10})$$

The right-hand side of (F.10) is a strictly increasing function of  $\hat{m}$ . From (F.8) and (F.9), we can verify that  $q'_0(\hat{m}) < 0$ ,  $b'_0(\hat{m}) < 0$ ,  $q_0(0) = \infty$ ,  $b_0(0) = 1$ ,  $q_0(\infty) = 0$  and  $\mu(b_0(\infty)) > 0$ . With these properties and the maintained assumptions on the function  $\mu(b)$ , we can verify that the left-hand side of (F.10) is a strictly decreasing function of  $\hat{m}$  and that there is a unique solution to (F.10) for the number  $\hat{m}$ . This solution, denoted as  $m_c$ , is the one used in (4.6). **QED**

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