Review of Economic Studies (2018) **85**, 892–928 doi:10.1093/restud/rdx048

Advance access publication 18 August 2017

The (Q,S,s) Pricing Rule

KENNETH BURDETT
University of Pennsylvania
and
GUIDO MENZIO
University of Pennsylvania and NBER

First version received April 2014*; Editorial decision August* 2017*; Accepted August* 2017 (*Eds.*)

We introduce menu costs in the search-theoretic model of imperfect competition of Burdett and Judd. When menu costs are not too large, the equilibrium is such that sellers follow a (Q,S,s) pricing rule. According to the rule, a seller lets inflation erode the real value of its nominal price until it reaches some point s. Then, the seller pays the menu cost and resets the real value of its nominal price to a point randomly drawn from a distribution with support $[S,Q]$, where $s<S<Q$. A (Q,S,s) equilibrium differs with respect to a standard (S,s) equilibrium: (1) in a (Q,S,s) equilibrium, sellers sometimes keep their nominal price constant to avoid paying the menu cost, other times because they are indifferent to changes in the real value of their price. An exploratory calibration reveals that menu costs account less than half of the observed duration of nominal prices. (2) in a (Q,S,s) equilibrium, higher inflation leads to higher real prices, as sellers pass onto buyers the cost of more frequent price adjustments, and to lower welfare.

Key words: Search frictions, Menu costs, Sticky prices.

JEL Codes: D11, D21, D43, E32

1. INTRODUCTION

It is a well-documented fact that sellers leave the nominal price of their goods unchanged for months in the face of a continuously increasing aggregate price level (see, *e.g.*, Klenow and Kryvtsov, 2008 or Nakamura and Steinsson, 2008). The standard explanation for nominal price stickiness is that sellers have to pay a menu cost to change their price (see, *e.g.*, Sheshinski and Weiss, 1977 or Caplin and Leahy, 1997). In the presence of such cost, sellers find it optimal to follow an (S,s) pricing rule: they let inflation erode the real value of their nominal price until it reaches some point s and, then, they pay the menu cost and reset their nominal price so that its real value is some S, with $S>s$.

Head *et al*. (2012) recently advanced an alternative explanation for nominal price stickiness. In the presence of search frictions, the equilibrium distribution of real prices posted by sellers is non-degenerate. Sellers are indifferent between posting any two real prices on the support of the equilibrium distribution. If the seller posts a relatively high price, it enjoys a higher profit per trade but it faces a lower probability of trade. If the seller posts a relatively low price, it enjoys a lower profit per trade but it faces a higher probability of trade. For this reason, sellers are also willing to keep their nominal price unchanged until inflation pushes its real value outside of the support of the equilibrium price distribution. Only then sellers have to change their nominal price.

The role played by nominal price stickiness in the transmission of monetary shocks to the real side of the economy crucially depends on whether prices are sticky because of menu costs or search frictions. Indeed, if nominal prices are sticky due to menu costs, an unexpected change in the quantity of money will in general have an effect on the distribution of real prices–as not every seller will find it worthwhile to pay the menu cost and adjust its nominal price–and, in turn, it will have an effect on consumption and production. In contrast, if nominal prices are sticky due to search frictions, an unexpected change in the quantity of money will have no effect on the distribution of real prices. Some sellers might keep their nominal price unchanged–as long as their price is still on the support of the equilibrium distribution–but the overall nominal price distribution will fully adjust to the monetary shock and will not affect consumption or production.

The observations above suggest that, to understand how monetary policy works, we first need to understand why nominal prices are sticky. This is the goal of our article. To accomplish this goal, we develop a model of the product market in which buyers and sellers face search frictions in trade, and sellers face menu costs in adjusting their nominal prices. We calibrate the model using data on the duration of nominal prices at individual sellers and on the dispersion of prices across different sellers in the same market. We then use the calibrated model to measure the relative contribution of menu costs and search frictions to the nominal stickiness of prices that we see in the data.

We consider a product market populated by a continuum of buyers and sellers who, respectively, demand and supply an indivisible good. Buyers search the market for sellers. The outcome of the search process is such that some buyers are captive—in the sense that they find only one seller—and some buyers are non-captive—in the sense that they find multiple sellers. Sellers cannot discriminate between the two types of buyers and, hence, they post a single price. The price is set in nominal terms and can only be changed by paying a menu cost. Essentially, our model is a dynamic version of Burdett and Judd (1983) with menu costs.

When menu costs are zero, the unique equilibrium of the model is such that different sellers post different prices for the same good (see Burdett and Judd, 1983). Price dispersion emerges because of the coexistence of non-captive buyers—which guarantee that an individual seller can increase its profit by undercutting any common price above marginal cost—and captive buyers—which guarantees that the undercutting process cannot drive a common price down to marginal cost. When menu costs are positive but small, the unique equilibrium of the model is such that sellers follow a (Q,S,s) pricing rule. According to this rule, the seller lets inflation erode the real value of its nominal price until it reaches some point s. Then, the seller pays the menu cost and changes its nominal price so that the real value of the new price is a random draw from a distribution with some support $[S,Q]$, with $Q>S>s$. When menu costs are large, the equilibrium is such that sellers follow a standard (S,s) rule. The key finding is that, as long as the menu costs are small enough, the unique equilibrium is such that sellers follow a (Q,S,s) rule rather than a standard (S,s) rule. We show that the forces that rule out the existence of an (S,s) equilibrium when menu costs are small are the same forces that rule out a one-price equilibrium when there are no menu costs.

A (Q,S,s) equilibrium looks like a hybrid between the equilibrium in a standard search model of price dispersion (see, *e.g.*, Burdett and Judd, 1983) and the equilibrium in a standard menu cost model of price stickiness (see, *e.g.*, Sheshinski and Weiss, 1977 and Bénabou, 1988). The seller's present value of profits is maximized for all real prices between S and Q. Also, over the interval between S and Q, the distribution of prices across sellers is the one that keeps the seller's flow profit constant. These are the key features of equilibrium in Burdett and Judd (1983). The seller's present value of profits increases monotonically for all real prices between s and S. Also, over the interval between s and S, the distribution of prices across sellers is log-uniform. These are the key features of equilibrium in Sheshinski and Weiss (1977) and Bénabou (1988). The natural combination of properties of search models and menu cost models emerges from a rather surprising feature of individual behaviour: When they pay the menu cost, sellers randomize over the real value of their new nominal price.

We establish two substantive differences between a (Q,S,s) equilibrium and a standard (S,s) equilibrium. The first difference is related to the cause of price stickiness. In a (Q,S,s) equilibrium, both menu costs and search frictions contribute to stickiness. As inflation erodes the real value of a nominal price from Q to S, the seller's profit remains constant and maximized. Here, the seller would not want to adjust its nominal price even if it could do it for free. As inflation further erodes the real value of the nominal price from S to s, the seller's profit declines. Here, the seller would want to adjust its nominal price, but it does not in order to avoid paying the menu cost. Overall, menu costs cause only part of the stickiness of a nominal price. The other part is caused by search frictions—specifically search frictions such that some buyers are captive and some are not—which create an interval of prices over which the seller's present value of profits is constant and maximized. In contrast, in an (S,s) equilibrium, the seller's present value of profits is only maximized at the highest price and all stickiness is due to menu costs.

The second substantive difference is related to the welfare effect of inflation. In a (Q,S,s) equilibrium, inflation has no effect on the sellers' maximum profit. This observation implies that inflation has no effect on the measure of sellers entering the market and, for any generic matching function, on the buyers' probability of meeting sellers. The observation that inflation does not affect the sellers' maximum profit also implies that, while inflation increases the resources spent by sellers to adjust their nominal price, this increase in expenditures is passed to the buyers via higher prices. Overall, higher inflation leaves the lifetime utility of sellers unaffected and unambiguously lowers the lifetime utility of buyers, thus causing welfare to fall. In contrast, in an (S,s) equilibrium, inflation lowers the sellers' maximum profit. In turn, this leads to a decline in the number of sellers in the market, in the buyers' probability of trade, and to lower prices. As shown in Bénabou (1988, 1992) and Diamond (1993), the latter effect may result in welfare being maximized at a strictly positive rate of inflation.

In the last part of the article, we carry out an exploratory calibration of the theory. We calibrate the model using data on the average duration of the price of a particular good at a particular store (from Nakamura and Steinsson, 2008), as well as data on the dispersion of the price of a particular good across different stores at a particular time and in a particular market (from Kaplan *et al.*, 2016). In all of our calibrations, we find that the equilibrium is such that sellers follow a (Q,S,s) rule. This finding implies that welfare is, at least locally, decreasing in inflation. We also find that both search frictions and menu costs contribute to some extent to the observed stickiness of nominal prices, although search frictions are relatively more important. This finding suggests that theories of price stickiness that abstract from search frictions—e.g. Dotsey *et al.* (1999)–are likely to overestimate the magnitude of menu costs and, in turn, the importance of nominal price rigidities as a channel of transmission of monetary policy shocks to the real side of the economy. Similarly, theories of price stickiness that abstract from menu costs—*e.g.* Head *et al.* (2012)—are likely to underestimate the importance of nominal price rigidities as a channel of transmission of monetary policy shocks. While these quantitative findings have to be taken with a grain of salt as our theory is too stark to capture all features of the data, they still provide useful information as back-of-the-envelope calculations.

The papers makes both theoretical and empirical contributions. On the theory side, the main contribution is to prove that—in a search-theoretic model of price dispersion—sellers follow a (Q,S,s) rule when menu costs are small. This is a new finding. Indeed, the literature on menu cost models universally finds that sellers follow an (S,s) pricing rule, or some analogous pricing rule with deterministic exit and reset prices.[1] Sheshinski and Weiss (1977), Caplin and Spulber (1987) and Caplin and Leahy (1997, 2010) prove the optimality of an (S,s) rule for a monopolist facing an exogenous downward-sloping demand curve. Dotsey *et al*. (1999) and Golosov and Lucas (2007) prove the optimality of an (S,s) rule in an equilibrium model where a continuum of monopolistic competitors face an endogenous demand curve derived from the buyers' Dixit-Stiglitz utility function. Bénabou (1988, 1992) shows that (S,s) rules are optimal in the search model of Diamond (1971) where all buyers are captive. Midrigan (2011) and Alvarez and Lippi (2014) generalize the notion of an (S,s) rule for sellers of multiple goods. The (Q,S,s) pricing rule is not only interesting because it is theoretically novel, but also because—we believe—it will arise in most product markets where the nature of competition between sellers is such that price dispersion must emerge in equilibrium (see, *e.g*., Prescott, 1975, Eden, 1994 or Menzio and Trachter, 2016). This type of markets seems the most relevant given how systematically and how widely the Law of One Price fails.

On the empirical side, the main contribution of the article is to attempt to measure how much of the observed stickiness of nominal prices is due to menu costs and how much to search frictions. We find that both menu costs and search frictions are responsible for some price stickiness. Indeed, if all stickiness was due to menu costs, the theory would imply much less price dispersion than in the data. Conversely, if all stickiness was due to search frictions, the theory would imply more price dispersion than in the data. A combination of menu costs and search frictions is required to simultaneously match empirical measures of price stickiness and price dispersion. Substantively, the result is important because price stickiness is a channel of transmission of monetary policy shocks to the real side of the economy only to the extent that such stickiness is caused by menu costs. Methodologically, the result is interesting because it shows that proper estimation of menu costs and search frictions requires taking simultaneously into account both dispersion and duration of prices.

2. ENVIRONMENT

We study a dynamic and monetary version of a model of imperfect competition in the spirit of Butters (1977), Varian (1980) and Burdett and Judd (1983). The market for an indivisible good is populated by a continuum of identical sellers with measure 1. Each seller maximizes the present value of real profits, discounted at the rate $r > 0$. Each seller produces the good at a constant marginal cost, which, for the sake of simplicity, we assume to be zero. Each seller posts a nominal price d for the good, which can only be changed by paying the real cost c, with $c > 0$.

The market is also populated by a continuum of identical buyers. In particular, during each interval of time of length dt, a measure bdt of buyers enters the market. A buyer comes into contact with one seller with probability α and with two sellers with probability $1-\alpha$, where $\alpha \in (0,1)$. We refer to a buyer who contacts only one seller as captive, and to a buyer who contacts two sellers as non-captive. Then, the buyer observes the nominal prices posted by the contacted sellers and decides whether and where to purchase a unit of the good. If the buyer purchases the good at the

1. Bénabou (1989) is one exception. He studies the pricing problem of a monopolist facing menu costs and a population of heterogeneous buyers. Some buyers have the ability to store the good. Other buyers must consume the good right away. The presence of buyers with storage ability gives the seller an incentive to randomize over the timing of a price adjustment. This pricing rule has the same flavor of our (Q,S,s) pricing rule in the case of deflation (see Section 4). Yet, the economic forces behind the randomization are different. In Bénabou (1989), the seller needs to randomize to prevent the buyers with storage ability from timing their purchases right before a price adjustment takes place. In our model, sellers randomize because, if they played pure symmetric strategies, there would not be enough price dispersion to maintain equilibrium.

nominal price d, he obtains a utility of $Q-\mu(t)d$, where $\mu(t)$ is the utility value of a dollar at date t and $Q>0$ is the buyer's valuation of the good. If the buyer does not purchase the good, he obtains a reservation utility, which we normalize to zero. Whether the buyer purchases the good or not, he exits the market.

The utility value of a dollar declines at the constant rate π. Therefore, if a nominal price remains unchanged during an interval of time of length dt, the real value of the price falls by $\exp(-\pi\, dt)$. In this article, we do not describe the demand and supply of dollars. It would, however, be straightforward to embed our model into either a standard cash-in-advance framework (see, e.g., Lucas and Stokey, 1987) or in a standard money-search framework (see, e.g., Lagos and Wright, 2005) and show that, in a stationary equilibrium, the depreciation rate π would be equal to the growth rate of the money supply.

Even without inflation and menu costs, the equilibrium of the model features a non-degenerate distribution of prices. The logic behind this result is clear. If all sellers post the same price, an individual seller can increase its profits by charging a slightly lower price and sell not only to the contacted buyers who are captive, but also to the contacted buyers who are not captive. This Bertrand-like process of undercutting cannot push all prices down to the marginal cost. In fact, if all sellers post a price equal to the marginal cost, an individual seller can increase its profits by charging the reservation price Q and sell only to the contacted buyers who are captive. Thus, in equilibrium, there must be price dispersion.

There are two differences between our model and Burdett and Judd (1983) (henceforth, BJ83). First, in our model sellers post nominal prices that can only be changed by paying a menu cost, while in BJ83 sellers post real prices that can be freely changed in every period. This difference is important because it implies that in our model the problem of the seller is dynamic, while in BJ83 it is static. Second, in our model the fraction of buyers meeting one and two sellers is exogenous, while in BJ83 it is an endogenous outcome of buyers' optimization. We believe that our results would generalize to an environment where buyers' search intensity is endogenous.[2]

There are also two differences between our model and Bénabou (1988) (henceforth, B88). First, in our model there are some buyers who are in contact with one seller and some who are in contact with multiple sellers, while in B88 all buyers are temporarily captive. This difference is important because it implies that, even without menu costs, the equilibrium of our model features price dispersion, while in B88 every seller would charge the monopoly price (as in Diamond, 1971). Second, in our model buyers have to leave the market after they search today, while in B88 they can choose to stay in the market and search again tomorrow. Our results would generalize to an environment where buyers are allowed to search repeatedly.[3] Therefore, the reader can think of B88 as a special case of our model in which $\alpha=1$.

3. INFLATION

In this section, we study equilibrium in the case of a positive rate of inflation. In Section 3.1, we formally define a (Q,S,s) and an (S,s) equilibrium. In Section 3.2, we establish conditions for the existence of a (Q,S,s) equilibrium. In Section 3.3, we establish conditions for the existence of

2. Consider a model where buyers can choose whether to search once or twice. Clearly, given the appropriate choice of the distribution of search costs across buyers, the equilibrium of our model is also an equilibrium of the model with endogenous search intensity.

3. Consider a model where buyers can search repeatedly and have a valuation of the good $Z\geq Q$ and a discount factor ρ. Given the appropriate choice of Z and ρ, the equilibrium of our model is also an equilibrium of the model with repeated search. In the model with repeated search, Q does not represent the buyer's valuation of the good, but the buyer's reservation price.

an (S,s) equilibrium. Our main finding is that, when the menu cost is small enough, the unique equilibrium is such that sellers follow a (Q,S,s) pricing rule and that in such equilibrium both search frictions and menu costs contribute to price stickiness. In Section 3.4, we generalize our findings to a version of the model where the good is divisible.

3.1. *Definition of equilibrium*

We begin by defining a stationary (Q,S,s) equilibrium. In a (Q,S,s) equilibrium, each seller lets inflation erode the real value of its nominal price until it reaches some level s, with $s \in (0,Q)$. Then, the seller pays the menu cost and resets the nominal price so that its real value is a random draw from a distribution with support $[S,Q]$, with $S \in (s,Q)$. We denote as F the distribution of the real value of nominal prices across all sellers in the market and as G the distribution of the real value of new nominal prices across sellers who just paid the menu cost, where both F and G are equilibrium objects.[4] We also find it useful to denote as $p(t)$ the real value of a nominal price that, t units of time ago, would have had a real value of Q, i.e. $p(t)=Qe^{-\pi t}$. We then let T_1 denote the amount of time it takes for inflation to lower the real value of a nominal price from Q to S, and T_2 denote the amount of time it takes for inflation to lower the real value of a nominal price from S to s, that is $T_1=\log(Q/S)/\pi$ and $T_2=\log(S/s)/\pi$.

Consider a seller whose current nominal price has a real value of $p(t)$. The present value of the seller's profits, $V(t)$, is given by

$$V(t)=\max_T \int_t^T e^{-r(x-t)}R(p(x))\,dx+e^{-r(T-t)}\left(V^*-c\right), \tag{1}$$

where

$$R(p(x))=[b\alpha+2b(1-\alpha)(1-F(p(x)))]p(x). \tag{2}$$

The above expressions are easy to understand. After $x-t$ units of time, the seller's nominal price has real value of $p(x)$ and the seller enjoys the flow profit $R(p(x))$. After $T-t$ units of time, the seller pays the menu cost c, resets the nominal price and attains the maximized present value of profits V^*. The seller's flow profit $R(p(x))$ is given by the sum of two terms. The seller meets a captive buyer at the rate $b\alpha$. Conditional on meeting a captive buyer, the seller trades with probability 1 and enjoys a profit of $p(x)$. The seller meets a non-captive buyer at the rate $2b(1-\alpha)$. Conditional on meeting a non-captive buyer, the seller trades with probability $1-F(p(x))$ and enjoys a profit of $p(x)$.

The seller chooses when to pay the menu cost and reset its nominal price. From (1), it follows that the seller finds it optimal to pay the menu cost when the real value of its nominal price is equal to $s=p(T_1+T_2)$ only if

$$R(p(T_1+T_2))=r(V^*-c). \tag{3}$$

The above expression is intuitive. The seller's marginal benefit from waiting an additional instant to pay the menu cost is $R(p)$, that is the seller's flow profit. The seller's marginal cost from waiting an additional instant to pay the menu cost is $r(V^*-c)$, that is the annuitized value of paying the menu cost and then attaining the maximized present value of profits. The seller finds it optimal

4. Notice that F is must be a continuous function. Indeed, it is immediate to verify that there cannot be a positive measure of sellers with the same price p in a stationary equilibrium where all sellers follow the same (Q,S,s) rule.

to pay the menu cost only if the marginal benefit and the marginal cost of waiting are equated. Condition (3) is also sufficient if[5]

$$R(p(t)) \geq r(V^* - c),\ \forall t \in [0, T_1 + T_2]. \tag{4}$$

When it pays the menu cost, the seller chooses the real value of its new nominal price. The seller finds it optimal to choose any real price in the interval $[S, Q]$ if and only if the present value of its profits attains the value V^* for all prices in $[S, Q]$, and attains a value smaller than V^* for all prices in $[s, S]$.[6] That is,

$$V(t) = V^*, \forall t \in [0, T_1], \tag{5}$$

$$V(t) \leq V^*,\ \forall t \in [T_1, T_1 + T_2]. \tag{6}$$

We find it useful to rewrite condition (5) as

$$R(p(t)) = rV^*,\ \forall t \in [0, T_1], \tag{7}$$

and

$$\int_{T_1}^{T_1+T_2} e^{-r(x-T_1)} R(p(x))\, dx + e^{-rT_2}\left(V^* - c\right) = V^*. \tag{8}$$

Let us explain the two conditions above. Suppose that (5) holds. In this case, the present value of profits for a seller with a real price of $S = p(T_1)$ is equal to V^*. This is condition (8). Moreover, the derivative of the present value of profits with respect to the age of the seller's price is equal to 0 for all $t \in [0, T_1]$. Since $V'(t) = rV(t) - R(p(t))$ and $V(t) = V^*$, we obtain condition (7). Conversely, if (7) and (8) hold, $V(t) = V^*$ for all $t \in [0, T_1]$.

The cross-sectional distribution of prices across sellers, F, is stationary if and only if the measure of sellers whose real price enters the interval $[s, p]$ is equal to the measure of sellers whose real price exits the same interval $[s, p]$ during an arbitrarily short period of time of length dt. These inflow-outflow conditions are given by

$$F(e^{\pi\, dt} p) - F(p) = F(e^{\pi\, dt} s) - F(s),\ \forall p \in (s, S), \tag{9}$$

$$F(e^{\pi\, dt} p) - F(p) = \Big[F(e^{\pi\, dt} s) - F(s)\Big][1 - G(p)],\ \forall p \in (S, Q). \tag{10}$$

The left-hand side of (9) and (10) is the flow of sellers into the interval $[s, p]$. The inflow is given by the measure of sellers whose real price is between p and $pe^{\pi\, dt}$. In fact, each of these sellers starts with a price greater than p and, after dt units of time, has a real price smaller than p. The right-hand side of (9) and (10) is the flow of sellers out of the interval $[s, p]$. If p is smaller than S, the outflow is given by the measure of sellers whose real price is between s and $se^{\pi\, dt}$. Indeed, each of these sellers starts with a price smaller than p and, within the next dt units of time, pays the menu cost, chooses a real price in the interval $[S, Q]$ and ends with a price greater than p. If p is greater than S, the outflow is given by the measure of sellers who pay the menu cost within

5. To be precise, sufficiency also requires $R(p(t)) \leq r(V^* - c)$ for $t \geq T_1 + T_2$. This inequality is implied by the necessary condition $R(p(T_1 + T_2)) = r(V^* - c)$ because $R(p) < R(s)$ for all $p < s$.

6. To be precise, one would also have to check that $V(t) < V^*$ for all $t < 0$. This is trivially satisfied as a seller never finds it optimal to set the real value of its nominal price to some $p > Q$.

the next dt units of time, $F(e^{\pi dt}s)-F(s)$, times the probability that the new price is greater than p, $1-G(p)$.

Dividing by dt and taking the limit as $dt \to 0$, the inflow–outflow conditions become

$$F'(p)p = F'(s)s, \; \forall p \in (s,S), \tag{11}$$

$$F'(p)p = F'(s)s[1-G(p)], \; \forall p \in (S,Q). \tag{12}$$

Condition (11) is a differential equation for F over the interval (s,S). Similarly, condition (12) is a differential equation for F over the interval (S,Q). The boundary conditions associated to these differential equations are

$$F(s)=0, \; F(S-)=F(S+), \; F(Q)=1. \tag{13}$$

Intuitively, $F(s)=0$ as there are no sellers who let their price fall below s. Similarly, $F(Q)=1$ as there are no sellers who reset their real price above Q. Finally, $F(S-)=F(S+)$ as the cross-sectional distribution of real prices across sellers is continuous.[7]

We are now in the position to define a (Q,S,s) equilibrium.

Definition 1. *A stationary (Q,S,s) equilibrium is a cumulative distribution of prices* $F:[s,Q]\to[0,1]$*, a cumulative distribution of new prices* $G:[S,Q]\to[0,1]$*, a pair of prices* (s,S) *with* $0<s<S<Q$*, and a maximized present value of seller's profits* V^* *that satisfy the optimality conditions (3), (4), (6), (7), (8) and the stationarity conditions (11)–(13).*

Next, we define a stationary (S,s) equilibrium. In an (S,s) equilibrium, each individual seller lets inflation erode the real value of its nominal price until it reaches some level $s\in(0,Q)$. Then, the seller pays the menu cost and adjusts the nominal price so that its real value is $S=Q$. Notice that one can think of an (S,s) equilibrium as a version of a (Q,S,s) equilibrium in which the lower bound S on the distribution of new prices is equal to the upper bound Q on the distribution of new prices.

Using the above insight, we can define an (S,s) equilibrium as follows.

Definition 2. *A stationary (S,s) equilibrium is a cumulative distribution of prices* $F:[s,Q]\to[0,1]$*, a pair of prices* (s,S) *with* $0<s<S=Q$*, and a maximized present value of seller's profits* V^* *that satisfy the optimality conditions (3), (4), (6), (8) and the stationarity conditions (11), (13).*

The (Q,S,s) and the (S,s) equilibria defined above are the only possible equilibria within the set of stationary equilibria in which sellers follow a symmetric pricing strategy such that: (1) the seller pays the menu cost when the real value of the nominal price reaches a point randomly drawn from distribution with support $[A,B]$; (2) the seller resets the nominal price so that its real value is randomly drawn from some distribution with support $[C,D]$. Indeed, it is easy to show that, in any equilibrium with inflation, we must have $A=B$, and $D=Q$.[8]

7. As pointed out by one of the referees, one can obtain (11)–(13) as straightfoward applications of the Kolmogorov Forward Equation.

8. First, let us explain why $D=Q$. Suppose $D<Q$. The necessary condition for optimality of the seller's new price implies $rV^*=R(D)=b\alpha D$. Since $R(p)=b\alpha p>R(D)$ for all $p\in(D,Q]$, the seller would be strictly better off resetting

3.2. *(Q,S,s) Equilibrium*

We now focus on the (Q,S,s) equilibrium. First, we solve for the pricing strategy of sellers and the stationary distribution of prices, and we identify a necessary and sufficient condition for the existence of a (Q,S,s) equilibrium. Second, we illustrate in details the properties of a (Q,S,s) equilibrium. Finally, we prove that the existence condition for a (Q,S,s) equilibrium is satisfied when either the menu cost and/or the inflation rate are not too high.

3.2.1. Existence. Consider the equilibrium condition (7), which guarantees that the derivative of the seller's present value of profits $V(t)$ with respect to the age t of the price is equal to zero for all $t \in [0, T_1]$. For $t = 0$, (7) states that the flow profit for a seller with a real price of Q is equal to the annuitized maximum present value of the seller's profits, that is $R(Q) = rV^*$. Since a seller with a real price of Q only trades with captive buyers, it enjoys a flow profit of $R(Q) = b\alpha Q$. Using this observation, we can solve $R(Q) = rV^*$ with respect to V^* and find

$$V^* = b\alpha Q \big/ r. \tag{14}$$

The equilibrium value for V^* has a simple interpretation. The maximized present value of the sellers' profits is equal to the value that would be attained by a hypothetical seller who always charges the buyer's reservation price Q and trades only with captive buyers.

Consider again the equilibrium condition (7). For $t \in [0, T_1]$, (7) states that the flow profit for a seller with a real price of $p \in [S, Q]$ is equal to the annuitized maximum present value of the seller's profits, that is $R(p) = rV^*$. Since a seller with a real price of p enjoys a flow profit $R(p)$ equal to $b[\alpha + 2(1-\alpha)(1-F(p))]p$ and the maximum present value of the seller's profits V^* is equal to (14), we can solve $R(p) = rV^*$ with respect to the price distribution F. We then find that

$$F(p) = 1 - \frac{\alpha}{2(1-\alpha)} \frac{Q-p}{p}, \ \forall p \in [S, Q]. \tag{15}$$

Notice that, over the interval $[S, Q]$, the equilibrium price distribution is the same as in BJ83. This finding is easy to understand. The price distribution equalizes the seller's flow profit for all prices in the interval $[S, Q]$. The price distribution plays exactly the same role in BJ83 and in many other search-theoretic models of price dispersion. Hence, over the interval $[S, Q]$, the price distribution in our model has the same shape as in BJ83.

Next consider the stationarity condition (11), which is an equation for the derivative of the price distribution F over the interval (s, S). Integrating (11) and using the boundary conditions $F(s) = 0$ and $F(S-) = F(S+)$, we find that

$$F(p) = \left[1 - \frac{\alpha}{2(1-\alpha)} \frac{Q-S}{S}\right] \frac{\log p - \log s}{\log S - \log s}, \ \forall p \in [s, S]. \tag{16}$$

Notice that, over the interval $[s, S]$, the equilibrium price distribution is the same as in B88. The finding is easy to understand. Sellers enter the interval $[s, S]$ from the top, they travel through

its price to Q rather than D. Suppose $D > Q$. The necessary condition for optimality of the seller's new price implies $rV^* = R(D) = 0$. Then $V^* - c < 0$, the seller would never pay the menu cost and there is no stationary equilibrium. Next, let us explain why $A = B$. Suppose $A < B$. The necessary condition for optimality of the seller's exit price implies $r(V^* - c) = R(p)$ for all $p \in [A, B]$. Solving this indifference condition with respect to F gives $F(p) = \frac{2-\alpha}{2(1-\alpha)} \frac{p-A}{p}$ for all $p \in (A, B)$. In turn, this implies $F'(p_1)p_1 > F'(p_2)p_2$ for all $p_1, p_2 \in (A, B)$ with $p_1 < p_2$. However, the measure of sellers with a price between p_1 and $p_1 + \epsilon$ cannot be greater than the measure of sellers who T units of time before had a price between $p_1 e^{\pi T}$ and $(p_1 + \epsilon)e^{\pi T}$, with $T = \log(p2/p1)/\pi$. This is because sellers can reach the price p_1 only if they has a price of p_2 T units of time ago. Taking the limit for $\epsilon \to 0$, the inequality becomes $F'(p_1)p_1 < F'(p_2)p_2$ for all $p_1, p_2 \in (A, B)$ with $p_1 < p_2$.

the interval at the rate π, and they exit the interval from the bottom. This is the same pattern followed by sellers in B88 and in any model in which sellers follow an (S,s) rule. Since the log-uniform distribution is the only stationary distribution consistent with this type of motion, the price distribution in our model must have the same shape over the interval $[s,S]$ as in B88.

Now consider the stationarity condition (12), which is an equation for the derivative of the price distribution F over the interval (S,Q). Using the fact that the price distribution F is given by (15), we can solve for (12) with respect to the distribution of new prices G and find

$$G(p)=1-\left(1-\frac{\alpha}{2(1-\alpha)}\frac{Q-S}{S}\right)^{-1}\frac{\alpha\log(S/s)Q}{2(1-\alpha)p},\ \forall p\in(S,Q). \tag{17}$$

The derivation above makes it clear that the role of the distribution of new prices G is to generate the cross-sectional distribution of prices F that keeps the seller's profit constant over the interval $[S,Q]$ and, hence, makes sellers indifferent between resetting their price anywhere in the interval $[S,Q]$. One can easily see from (17) that, to fulfill its role, the distribution G must have a mass point of measure $\chi(S)=G(S)$ at S and a mass point of measure $\chi(Q)=1-G(Q-)$, where $\chi(S)$ and $\chi(Q)$ are given by

$$\chi(S)=1-\left(1-\frac{\alpha}{2(1-\alpha)}\frac{Q-S}{S}\right)^{-1}\frac{\alpha\log(S/s)Q}{2(1-\alpha)S}, \tag{18}$$

$$\chi(Q)=\left(1-\frac{\alpha}{2(1-\alpha)}\frac{Q-S}{S}\right)^{-1}\frac{\alpha\log(S/s)}{2(1-\alpha)}. \tag{19}$$

Finally, we need to solve for the equilibrium prices s and S. The optimality condition (3) states that the flow profit for a seller with a real price of s is equal to the annuity value of paying the menu cost, adjusting the nominal price, and enjoying the maximized present value of profits. That is, $R(s)=r(V^*-c)$. Since a seller with a real price of s trades with all the buyers it meets, $R(s)=b(2-\alpha)s$. Using this observation, we can solve $R(s)=r(V^*-c)$ with respect to s and find

$$s=\frac{\alpha Q-rc/b}{2-\alpha}. \tag{20}$$

The optimality condition (8) states that a seller with a real price of S must attain the maximized present value of profits V^*. After substituting out F, G and V^* and solving the integral, we can rewrite (8) as

$$\begin{aligned}\varphi(S)\equiv&\left[\frac{1-e^{-(r+\pi)T_2(S)}(1+(r+\pi)T_2(S))}{(r+\pi)^2}\right]\frac{(2-\alpha)S-\alpha Q}{T_2(S)}\\&+\left[\frac{1-e^{-(r+\pi)T_2(S)}}{r+\pi}\right]\alpha Q-\left[\frac{1-e^{-rT_2(S)}}{r}\right]\alpha Q-e^{-rT_2(S)}c/b=0,\end{aligned} \tag{21}$$

where

$$T_2(S)\equiv\frac{\log(S/s)}{\pi},\quad s=\frac{\alpha Q-rc/b}{2-\alpha}.$$

Clearly, a (Q,S,s) equilibrium exists only if the equation $\varphi(S)$ admits a solution for some S in the interval (s,Q). It is easy to verify that $\varphi(S)$ is strictly negative for all $S\in[s,\alpha Q/(2-\alpha)]$ and it is

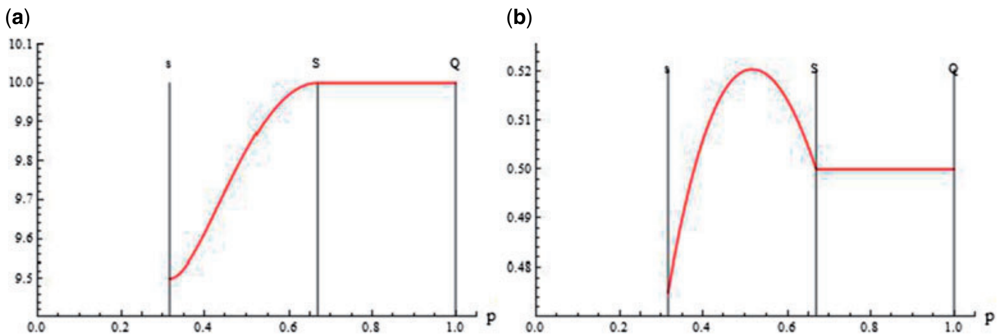

FIGURE 1

Present value of profits and flow profit. (a) Present value; (b) Flow profit

Notes: The present value of profits V and the flow profit R as functions of the real price p given the parameters $\alpha = 1/2$, $c = 1/2$, $b = 1$, $Q = 1$, $r = 5\%$ and $\pi = 3\%$.

strictly increasing for all $S \in [s, Q]$. Therefore, a (Q,S,s) equilibrium exists only if the equation $\varphi(Q) > 0$. As it turns out, $\varphi(Q) > 0$ is also a sufficient condition for the existence of a (Q,S,s) equilibrium.

The next theorem summarizes our findings, and provides a characterization of the seller's value function, $V(t)$, and flow payoff function, $R(p)$.

Theorem 1. *A (Q,S,s) equilibrium exists iff* $\varphi(Q) > 0$. *If a* (Q, S, s) *equilibrium exists, it is unique and: (a)* $V'(t) = 0$ *for all* $t \in (0, T_1)$ *and* $V'(t) < 0$ *for all* $t \in (T_1, T_1 + T_2)$*; (b)* $\hat{R}'(t) = 0$ *for all* $t \in (0, T_1)$*,* $\hat{R}'(t) > 0$ *for all* $t \in (T_1, \hat{T})$ *and* $\hat{R}'(t) < 0$ *for all* $t \in (T, T_1 + T_2)$*, where* $\hat{R}(t) \equiv R(p(t))$ *and* $\hat{T} \in (T_1, T_1 + T_2)$*.*

Proof. In Appendix A. ‖

3.2.2. Characterization. Figures 1 and 2 illustrate the qualitative features of a (Q,S,s) equilibrium. Figure 1a plots the seller's present value of profits as a function of the real value of the seller's price. Like in a standard search-theoretic model of price dispersion (*e.g.* BJ83), the seller's present value of profits remains equal to its maximum V^* as the real value of the nominal price goes from Q to S. Like in a standard menu cost model[9] (*e.g.* B88), the seller's present value of profits declines monotonically from V^* to $V^* - c$ as the real value of the nominal price falls from S to s. When the real value of the nominal price reaches s, the seller finds it optimal to pay the menu cost and reset its price.

When the seller pays the menu cost, it is indifferent between resetting its nominal price to any real value between S and Q. This property of the equilibrium may seem puzzling, as the seller would have to pay the menu cost less frequently if it were to reset the real price to Q rather than to, say, S. The solution to the puzzle is contained in Figure 1b, which plots the seller's flow profit as a function of the real price. As the real price falls from Q to S, the seller's flow profit is constant and equal to rV^*. As the real price falls below S, the seller's flow profit increases, reaches a maximum, and then falls to $r(V^* - c)$. Thus, if the seller resets its real price to Q rather than to some lower value, it pays the menu cost less frequently but it also enjoys the highest flow

9. With standard menu cost model we refer to any model in which sellers follow an (S,s) pricing rule.

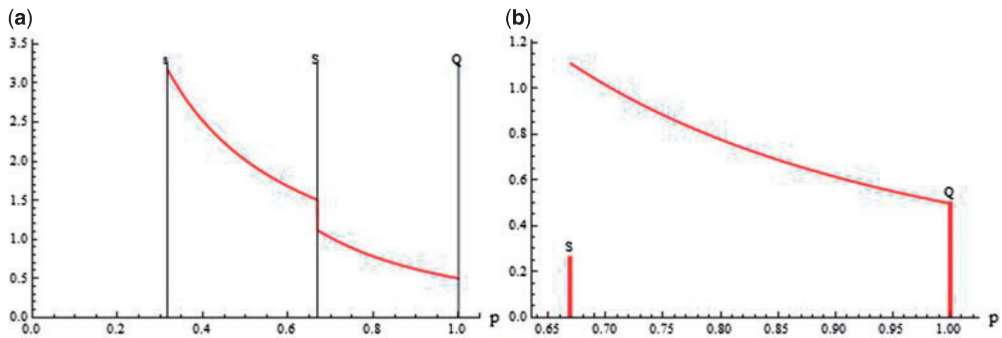

FIGURE 2

Equilibrium price distributions. (a) Price distribution; (b) New price distribution

Notes: The densities of the equilibrium distribution of prices and of the equilibrium distribution of new prices given the parameters $\alpha = 1/2$, $c = 1/2$, $b = 1$, $Q = 1$, $r = 5\%$ and $\pi = 3\%$.

profit less frequently. The two effects exactly cancel each other and, for this reason, the seller is indifferent between resetting its real price to any value between S and Q.

Figure 2a plots the density F' of the equilibrium price distribution. Over the interval $[S,Q]$, the equilibrium price distribution is such that the seller's flow profit is constant, just as in a standard search-theoretic model of price dispersion. Over the interval $[s,S]$, the distribution is log-uniform, just as in a standard menu cost model. At the border between the two intervals (i.e. at price S), there is a discontinuity in the density of the equilibrium price distribution. In particular, the density to the right of S is discontinuously lower than to the left of S. This discontinuity explains why the seller's flow profit increases when the real price falls below S. Let us expand on this point. As its real price falls, the seller experiences an increase in the volume of trade that is proportional to the increase in the number of firms charging a price higher than the seller's (an increase equal to the density of the price distribution). As the real price falls from Q to S, the density of the price distribution is such that the increase in the seller's volume of trade exactly offsets the decline in the seller's profit margin. As the real price falls below S, the density of the price distribution jumps up and, hence, the increase in the seller's volume of trade more than offsets the decline in the seller's price. Thus, the seller's flow profit increases.

Figure 2b plots the density of the equilibrium distribution of new prices G. The support of the distribution is the interval $[S,Q]$. The distribution has mass points at S and Q, and is continuous everywhere else. The fact that the distribution of new prices G has a mass point at S explains why the density of the price distribution F has a discontinuity at S, why the seller's flow profit increases when its real price falls below S and, ultimately, why the seller is indifferent between resetting its price anywhere between S and Q. The fact that the distribution of new prices G has a mass point at Q explains why the density of the equilibrium price distribution F is strictly positive at S, which is necessary for the seller's flow profit to remain constant as the real value of its price falls below Q.

In a (Q,S,s) equilibrium, both search frictions and menu costs contribute to the stickiness of nominal prices. Consider a seller that just reset the real value of its nominal price to some $p \in [S,Q]$. The seller keeps this nominal price unchanged for $\log(p/s)/\pi$ units of time. During the first $\log(p/S)/\pi$ units of time in the life of the price, the seller's present value of profits remains constant at V^*. During this phase, the seller would not want to change its price even if it could do it for free. During the last $\log(S/s)/\pi$ units of time in the life of the price, the seller's profit is strictly smaller than V^*. During this phase, the seller would like to change its nominal price but it chooses not to in order to avoid paying the menu cost. Only the second phase of nominal price stickiness is caused by menu costs. The first phase is caused by the fact that, in a (Q,S,s) equilibrium, profits are maximized over the whole interval $[S,Q]$. As we explain in Section 3.3, the existence of a profit maximizing interval $[S,Q]$ is due to search frictions and, in particular, to a search process in which there are both captive and non-captive buyers. The fact that in a (Q,S,s) equilibrium both search frictions and menu costs contribute to nominal price stickiness is in contrast with a standard menu cost model—that is a model where sellers follow a standard (S,s) rule. In fact, in a standard menu cost model, profits are maximized only at the highest price in the distribution and, hence, all price stickiness is due to menu costs.

3.2.3. Comparative statics. We now want to understand how changes in the parameters of the model affect outcomes in a (Q,S,s) equilibrium. In particular, we are interested in the effect of c and π on nominal price stickiness and on the contribution of search frictions and menu costs to nominal price stickiness. As a simple measure of nominal price stickiness, we use T_1+T_2, which is the longest duration of a nominal price in equilibrium (*i.e.* the duration of a nominal price that had a real value of Q at the time it was chosen by the seller). As a measure of the contribution of search frictions to stickiness, we use $T_1/(T_1+T_2)$, which is the fraction of time that the nominal price with the longest duration spends in the region $[S,Q]$ created by search frictions. As a measure of the contribution of menu costs to stickiness, we use $T_2/(T_1+T_2)$, which is the fraction of time that the nominal price with the longest duration spends in the region $[s,S]$ created by menu costs.

In Appendix B, we prove that a (Q,S,s) equilibrium exists if and only if the menu cost c belongs to the interval $(0,\overline{c})$, where $\overline{c}$ is a strictly positive number that depends on the value of the other parameters. Over the interval $(0,\overline{c})$, an increase in the menu cost leads to a decrease in s and to an increase in S. Intuitively, s falls because the cost of deferring the adjustment of the nominal price decreases with c. Similarly, S increases because the benefit of adjusting the nominal price to a higher real value increases with c. Given the response of s and S, it follows that an increase in the menu cost leads to: an increase in price stickiness, as measured by T_1+T_2; a decline in the contribution of search frictions to stickiness, as measured by $T_1/(T_1+T_2)$; an increase in the contribution of menu costs to stickiness, as measured by $T_2/(T_1+T_2)$.

When the menu cost c approaches $\overline{c}$, the (Q,S,s) equilibrium converges to the equilibrium of a standard menu cost model (*e.g.* B88). In fact, for $c\to\overline{c}$, the lowest price in the distribution of new prices, S, converges to the highest price in the distribution of new prices, Q. Therefore, for $c\to\overline{c}$, all sellers reset their price to Q, the present value of the sellers' profits attains its maximum only at Q, and the entire cross-sectional price distribution is log-uniform. Clearly, for $c\to\overline{c}$, all of nominal price stickiness is caused by menu costs. When the menu cost c approaches 0, the (Q,S,s) equilibrium converges to the equilibrium of a standard search-theoretic model of price dispersion (*e.g.* BJ83). In fact, for $c\to 0$, the lowest price in the cross-sectional distribution of prices, s, converges to the lowest price in the distribution of new prices, S. Therefore, for $c\to 0$, every price on the support of the cross-sectional distribution of prices maximizes the present value of the seller's profits, and the cross-sectional price distribution is such that the seller's flow profit is the same at every price. Clearly, for $c\to 0$, there is nominal price stickiness (as sellers only adjust the nominal price when it reaches a real value of s) and it is all caused by search frictions.[10]

10. Head *et al*. (2012) analyse a version of our model without menu costs. They show that the equilibrium pins down the distribution of real prices F, but not the distribution of new real prices G or the pricing strategy of individual sellers. For $c\to 0$, the equilibrium F of our model is exactly the same as theirs. However, for $c\to 0$, the equilibrium of

In Appendix B, we also prove that a (Q,S,s) equilibrium exists if and only if the inflation rate π belongs to the interval $(0,\overline{\pi})$, where $\overline{\pi}$ is a strictly positive number that depends on the value of the other parameters. Over the interval $(0,\overline{\pi})$, an increase in inflation does not affect s, but it leads to an increase in S. Intuitively, s does not change because neither the marginal cost or the marginal benefit of deferring the adjustment of a nominal price depend on π. In contrast, S increases because the benefit of resetting the nominal price to a higher value increases with π. Moreover, we prove that an increase in inflation leads to: a decline in nominal price stickiness, as measured by T_1+T_2; a decline in the contribution of search frictions to price stickiness, as measured by $T_1/(T_1+T_2)$; an increase in the contribution of menu costs to nominal price stickiness, as measured by $T_2/(T_1+T_2)$. When π approaches $\overline{\pi}$, the (Q,S,s) equilibrium converges to the equilibrium of a standard menu cost model. When π approaches 0, the (Q,S,s) equilibrium does not have any special properties, except that the travelling times T_1 and T_2 go to infinity.

The comparative statics results are collected in the following theorem.

Theorem 2. *(i) A (Q,S,s) equilibrium exists iff $c\in(0,\overline{c})$, where $\overline{c}>0$ depends on other parameters. As c increases in $(0,\overline{c})$: (a) nominal price stickiness, T_1+T_2, increases; (b) the contribution of search frictions to price stickiness, $T_1/(T_1+T_2)$, falls; (c) the contribution of menu costs to price stickiness, $T_2/(T_1+T_2)$, increases. (ii) A (Q,S,s) equilibrium exists iff $\pi\in(0,\overline{\pi})$, where $\overline{\pi}>0$ depends on other parameters. As π increases in $(0,\overline{\pi})$: (a) nominal price stickiness, T_1+T_2, falls; (b) the contribution of search frictions to price stickiness, $T_1/(T_1+T_2)$, falls; (c) the contribution of menu costs to price stickiness, $T_2/(T_1+T_2)$, increases.*

Proof. In Appendix B. ||

3.3. *(S,s) Equilibrium*

We now focus on the (S,s) equilibrium. We first solve for the equilibrium objects when sellers follow an (S,s) pricing rule. We then derive a necessary and sufficient condition under which these objects constitute an (S,s) equilibrium. Finally, we prove that the necessary and sufficient condition for the existence of an (S,s) equilibrium is satisfied if and only if the menu cost is neither too small nor too large.

To solve for the equilibrium objects when sellers follow an (S,s) pricing rule, start with the stationarity condition (11). Using the fact that $S=Q$ and the boundary conditions $F(s)=0$ and $F(Q)=1$, we can integrate (11) and find that

$$F(p)=\frac{\log p-\log s}{\log Q-\log s},\ \forall p\in[s,Q]. \tag{22}$$

Second, consider the optimality condition (3), which states that $R(s)=r(V^*-c)$. Since $R(s)=b(2-\alpha)s$, we can solve the optimality condition with respect to s and find

$$s=\frac{r\left(V^*-c\right)}{b(2-\alpha)}. \tag{23}$$

our model uniquely pins down the distribution of new prices and the pricing strategy of individual sellers. Indeed, for $c\to 0$, $G(p)=(p-s)/p$ and an individual seller changes its nominal price only when its real value reaches $s=\alpha Q/(2-\alpha)$. In this sense, the limit of our model for $c\to 0$ provides a natural refinement for the indeterminate equilibrium objects in Head *et al*. (2012).

Third, consider the optimality condition (8), which states that a seller with a real price of S must attain the maximized present value of profits V^*. Using the fact that S is equal to Q, F is given by (22) and s is given by (23), we can rewrite (8) as one equation in the unknown V^*. Specifically, we can rewrite it as

$$\begin{aligned}\vartheta(V^*)\equiv\quad&\left[\frac{1-e^{-(r+\pi)T(V^*)}\big(1+(r+\pi)T(V^*)\big)}{(r+\pi)^2}\right]\frac{2b(1-\alpha)Q}{T(V^*)}\\&+\left[\frac{1-e^{-(r+\pi)T(V^*)}}{r+\pi}\right]b\alpha Q+e^{-rT(V^*)}\big(V^*-c\big)-V^*=0,\end{aligned}\tag{24}$$

where

$$T(V^*)\equiv\frac{1}{\pi}\log\left(\frac{b(2-\alpha)Q}{r(V^*-c)}\right).$$

Equations (22)–(24) characterize the equilibrium objects when sellers follow an (S,s) pricing rule. Theorem 3 shows that these objects constitute an (S,s) equilibrium if and only if V^* belongs to the interval $(c,b\alpha Q/r]$. It is easy to establish the necessity of $V^*\in(c,b\alpha Q/r]$. Indeed, if $V^*\leq c$, a seller would never find it optimal to pay the menu cost. In this case, there would be no stationary distribution of prices and, hence, no (S,s) equilibrium. If $V^*>b\alpha Q/r$, a seller who pays the menu cost would find it optimal to reset the real value of its nominal price below Q. Also in this case, there would be no (S,s) equilibrium. To establish the sufficiency of $V^*\in(c,b\alpha Q/r]$, we just need to verify that the seller cannot increase its profit by paying the menu cost before the real value of the nominal price reaches s—that is, we need to verify the optimality condition (4)—and that it cannot increase its profit by resetting the nominal price to a real value smaller than Q—that is, we need to verify the optimality condition (6).

Theorem 3. *An* (S,s) *equilibrium exists iff* $\vartheta(V^*)=0$ *for some* $V^*\in(c,b\alpha Q/r]$.

Proof. In Appendix C. ||

Theorem 4 shows that the condition for existence of an (S,s) equilibrium is satisfied if and only if the menu cost is neither too small nor too large. More specifically, the menu cost c must be in the interval $[c_\ell,c_h]$, where c_ℓ is strictly positive and c_h is strictly smaller than $b\alpha Q/r$. The theorem also shows that the interval $[c_\ell,c_h]$ contains the highest menu cost $\bar{c}$ for which a (Q,S,s) equilibrium exists.

Theorem 4. *An (S,s) equilibrium exists iff* $c\in[c_\ell,c_h]$. *The bounds* c_ℓ *and* c_h *depend on other parameters but are always such that* $0<c_\ell<c_h<b\alpha Q/r$ *and* $c_\ell\leq\bar{c}\leq c_h$.

Proof. In Appendix D. ||

Theorem 4 together with Theorem 2 implies that, when the menu cost is small enough, there is a unique equilibrium and in this equilibrium sellers follow a (Q,S,s) pricing rule. When the menu cost takes on intermediate values, there may be coexistence of a (Q,S,s) equilibrium and an (S,s) equilibrium. When the menu cost is high enough (but still lower than c_h), all equilibria are such that sellers follow an (S,s) pricing rule. When the menu cost is greater than c_h, sellers do not want to pay the menu cost and there is no stationary equilibrium.

The key result is that, when the menu cost is sufficiently small, the unique equilibrium is such that sellers follow a (Q,S,s) pricing rule. Let us provide some intuition for this result. Start with a menu cost c for which an (S,s) equilibrium does exist. When we lower c, the real price s at which sellers find it optimal to pay the menu cost increases—as the marginal cost of a price adjustment falls—and moves closer and closer to the buyer's reservation price Q. As a result, the equilibrium price distribution F becomes more and more compressed around the buyer's reservation price Q. At some point, the price distribution becomes so compressed that an individual seller no longer finds it optimal to reset the real value of its nominal price to Q. Instead, an individual seller finds it optimal to reset the real value of its nominal price to some $p<Q$. The seller finds this deviation profitable because, by resetting its price to p rather than Q, it can sell not only to the captive buyers, but also to the increasing number $2b(1-\alpha)(1-F(p))$ of non-captive buyers who are in contact with a seller charging a price between p and Q. Thus, when the menu is sufficiently low, an (S,s) equilibrium with $S=Q$ does not exists.

Could there exist an (S,s) equilibrium with $S<Q$? The answer is negative. Indeed, suppose that there exists an (S,s) equilibrium in which every seller resets the real value of its nominal price to some $S<Q$. In this equilibrium, the present value of the seller's profits at S, which is the maximum V^*, must be greater than at any price smaller than S. This is possible only if $V^* \leq b\alpha S/r$. However, if $V^* \leq b\alpha S/r$, an individual seller finds it optimal to reset the real value of its nominal price to Q rather than S. Intuitively, the seller finds the deviation profitable because, by resetting its price to Q rather than S, it still only sells to the captive buyers but it enjoys a higher profit margin. More precisely, by resetting its price to Q rather than S, the seller enjoys a period of length $\log(Q/S)/\pi$ during which its flow profit is strictly greater than $b\alpha S$. Since $b\alpha S \geq rV^*$, the present value of the seller's profits is higher if he resets its price to Q rather than S. Thus, an (S,s) equilibrium with $S<Q$ cannot exist.

Overall, when the menu cost is sufficiently low, there cannot be an (S,s) equilibrium in which all sellers reset their prices to Q because then an individual seller would want to deviate and reset its price to some $p<Q$. There cannot be an (S,s) equilibrium in which all sellers reset their prices to $S<Q$ because then an individual seller would want to deviate and reset its price to Q. The resolution of this tension is an equilibrium in which sellers randomize with respect to their reset price: a (Q,S,s) equilibrium. As it is clear from the discussion above, the economic force that rules out an (S,s) equilibrium and asks instead for a (Q,S,s) equilibrium is exactly the same force that, absent menu costs, rules out a unique price equilibrium and asks instead for an equilibrium with price dispersion. Namely, a search process in which the fraction α of captive buyers and the fraction $1-\alpha$ of non-captive buyers are both strictly positive.

3.4. *Divisible good*

While the analysis of equilibrium is simplest under the assumption that the good is indivisible and buyers demand a single unit, for some applications it may be more realistic or more useful to assume that the good is divisible and buyers demand multiple units. To this aim, we consider an environment in which buyers have the utility function $u(x,y)=x^{1-\gamma}/(1-\gamma)+y$ with $\gamma>1$, where x are units of the goods purchased in the market with search frictions and y are units of a numeraire good with a nominal price of $1/\mu(t)$. Buyers can either acquire the frictional good from a seller or they can home-produce it. In particular, buyers have access to a home-production technology that allows them to transform the numeraire good into the frictional good at the rate of Q to 1. The assumption, borrowed from Kaplan and Menzio (2016), pins down the buyer's reservation price.

Under these assumptions, a buyer who finds a lowest real price of $p \in [0,Q]$ purchases $p^{-1/\gamma}$ units of the frictional good. A buyer who finds a lowest price of $p>Q$ does not purchase the frictional good and instead home produces $Q^{-1/\gamma}$ units of it. The buyer's optimal purchasing strategies implies that a seller with a real price of $p \in [0,Q]$ enjoys a flow profit $R(p)$ equal to

$b[\alpha+2(1-\alpha)(1-F(p))]p^{\theta}$, with $\theta\equiv(\gamma-1)/\gamma$. A seller with a real price of $p>Q$ enjoys a flow profit of zero. Overall, the only difference between the divisible and the indivisible good model is the θ in the flow profit function.

In Appendix E, we generalize many of the results derived for the indivisible good model in Sections 3.2 and 3.3 to the case of a divisible good. In particular, we show that a (Q,S,s) equilibrium exists if and only if the menu cost is not too large, and it is such that both search frictions and menu costs contribute to price stickiness. We also show that an (S,s) equilibrium does not exist if the menu cost is small enough.

4. DEFLATION

In the previous section, we characterized the equilibrium set in the case of positive inflation. We found that, when menu costs are small enough, the unique equilibrium is such that sellers follow a (Q,S,s) pricing rule. In this section, we turn our attention to the case of negative inflation (*i.e.* deflation). We show that, also in this case, the equilibrium is such that sellers follow a (Q,S,s) pricing rule when menu costs are small. However, while in the case of inflation a (Q,S,s) rule involves sellers randomizing over the new price they set after paying the menu cost, in the case of deflation a (Q,S,s) rule involves sellers randomizing over the price at which they pay the menu cost.

4.1. *Definition of (Q,S,s) Equilibrium*

We begin by defining a (Q,S,s) equilibrium in the case of deflation. In a (Q,S,s) equilibrium with deflation, each individual seller lets the real value of its nominal price drift up until it reaches some point $S\in(0,Q)$. At this point, the seller pays the menu cost with probability $\chi(S)$. If the seller pays the menu cost, it resets the real value of its price to some $s\in(0,S)$. Otherwise, the seller lets the real value of its nominal price drift further. When the real value of the price is $p\in(S,Q)$, the seller pays the menu cost at the rate $h(p)$ and resets the price to s. When the real value of the nominal price reaches Q, the seller pays the menu cost with probability 1.[11] We denote as F the distribution of the real value of nominal prices across all sellers in the market. We find it useful to define $p(t)$ as $se^{-\pi t}$, T_1 as $-\log(S/s)/\pi$ and T_2 as $-\log(Q/S)/\pi$.

In a (Q,S,s) equilibrium the seller is willing to pay the menu cost when its price is anywhere between $S=p(T_1)$ and $Q=p(T_1+T_2)$. This is true only if

$$R(p(t))=r(V^*-c),\ \forall t\in[T_1,T_1+T_2]. \tag{25}$$

The left-hand side is the marginal benefit of waiting another instant to pay the menu cost and the right-hand side is the marginal cost. Condition (25) then states that the seller is willing to pay the menu cost anywhere over the interval $[S,Q]$ only if marginal cost and benefit are equated. The condition is also sufficient if [12]

$$R(p(t))\geq r(V^*-c),\ \forall t\in[0,T_1]. \tag{26}$$

11. The description of equilibrium imposes quite a bit of structure, in the sense that it assumes that sellers with a real price of S pay the menu cost with some positive probability $\chi(S)$ and that sellers with a real price of $p\in(S,Q)$ pay the menu cost at some rate $h(p)$. The assumption is without loss in generality, as it is straightforward to verify that there is no other randomization process that can make the seller's present value of profits constant over the interval $[S,Q]$.

12. To be precise, sufficiency also requires $R(p(t))\leq r(V^*-c)$ for $t\geq T_1+T_2$. This inequality is implied by the fact that $R(p)=0$ for all $p>Q$.

In a (Q,S,s) equilibrium, when the seller pays the menu cost, it wants to reset its value to $s=p(0)$. This is true only if

$$R(p(0))=rV^*. \tag{27}$$

If $R(p(0))>rV^*$, the seller would be better off choosing a nominal price with a real value smaller than s. If $R(p(0))<rV^*$, the seller would be better off choosing a nominal price with a real value greater than s. Only if $R(p(0))=rV^*$, the seller finds it optimal to choose a nominal price with a real value of s. Condition (27) is also sufficient if [13]

$$V(0) = V^*, \tag{28}$$

$$V(t) \leq V^*, \forall t \in [0,T_1]. \tag{29}$$

The distribution of prices across sellers, F, is stationary if and only if

$$-\pi F'(p)pdt=H(S)dt,\ \forall p\in(s,S), \tag{30}$$

$$-\pi F'(p)pdt=H(p)dt,\ \forall p\in(S,Q), \tag{31}$$

where

$$H(p)=\int_p^Q h(x)F'(x)dx-\pi F'(Q)Q-\mathbf{1}[p=S]\pi F'(S-)S\chi(S). \tag{32}$$

The left-hand side of (30)–(31) is the flow of sellers out of the interval $[s,p]$ over an arbitrarily short period of time of length dt. The right-hand side is the flow of sellers into the interval $[s,p]$. The boundary conditions associated to the differential equations (30) and (31) are

$$F(s)=0,\ F(S-)=F(S+),\ F(Q)=1. \tag{33}$$

We are now in the position to define a (Q,S,s) equilibrium.[14]

Definition 3. *A stationary (Q,S,s) equilibrium is a cumulative price distribution $F:[s,Q]\to[0,1]$, a cumulative price adjustment rate $H:[S,Q]\to R_+$, a pair of prices (s,S) with $0<s<S<Q$, and a maximized present value of seller's profits V^* that satisfy the optimality conditions (25)–(29) and the stationarity conditions (30)–(33).*

4.2. *Existence and properties of (Q,S,s) Equilibrium*

We now solve for the equilibrium objects when sellers follow a (Q,S,s) pricing rule and find necessary and sufficient conditions under which these objects constitute an equilibrium. For

13. To be precise, one would also have to check that $V(t)<V^*$ for all $t<0$. However, this is a direct implication of the necessary condition $rV(0)=R(p(0))$ and of the fact that $R(p)<R(s)$ for all $p<s$.

14. It is easy to verify that the (Q,S,s) equilibrium defined below and the associated (S,s) equilibrium are the only possible equilibria in the family of equilibria where sellers follow a symmetric pricing strategy such that: (1) the seller pays the menu cost only when the real value of the nominal price reaches a point randomly drawn from a distribution with support $[A,B]$; (2) the seller resets the nominal price so that its real value is randomly drawn from a distribution with support $[C,D]$.

$t = T_1 + T_2$, the optimality condition (25) implies $R(Q) = r(V^* - c)$. Since $R(Q) = b\alpha Q$, we can solve this equation with respect to V^* and find

$$V^* = b\alpha Q \big/ r + c. \tag{34}$$

For $t \in [T_1, T_1 + T_2]$, the optimality condition (25) implies $R(p) = r(V^* - c)$. Since $R(p)$ is given by $b[\alpha + 2(1-\alpha)(1-F(p))]p$ and V^* is given by (34), we can solve the equation with respect to F and find

$$F(p) = 1 - \frac{\alpha}{2(1-\alpha)} \frac{Q-p}{p}, \ \forall p \in [S, Q]. \tag{35}$$

The stationarity condition (30) is an equation for the derivative of the price distribution F over the interval (s, S). Integrating and using the boundary conditions $F(s) = 0$ and $F(S-) = F(S+)$, we find

$$F(p) = \left[1 - \frac{\alpha}{2(1-\alpha)} \frac{Q-S}{S}\right] \frac{\log p - \log s}{\log S - \log s}, \ \forall p \in [s, S]. \tag{36}$$

The stationarity condition (31) is an equation for the derivative of the price distribution F over the interval (S, Q). Using the fact that F is given by (35), we can solve (31) with respect to the rate $h(p)$ at which sellers pay the menu cost when their real price is $p \in (S, Q)$ and for the probability $\chi(S)$ at which sellers pay the menu cost when their real price is S. We find that the rate $h(p)$ is given by

$$h(p) = -\pi, \ \forall p \in (S, Q). \tag{37}$$

The probability $\chi(S)$ is given by

$$\chi(S) = 1 - \left[\frac{\alpha}{2(1-\alpha)} \frac{Q}{S}\right] \left[\left(1 - \frac{\alpha}{2(1-\alpha)} \frac{Q-S}{S}\right) \frac{1}{\log(S/s)}\right]^{-1}. \tag{38}$$

Finally, we need to solve for s and S. The optimality condition (27) states that $R(s) = rV^*$. Since $R(s) = b(2-\alpha)s$, we can solve the equation with respect to s and find

$$s = \frac{\alpha Q + rc/b}{2-\alpha}. \tag{39}$$

The optimality condition (28) states that the present value of profits for a seller with a real price of $s = p(0)$ is equal to V^*. After substituting out F and V^*, we can rewrite (28) as one equation with respect to S. That is,

$$\begin{aligned} \varphi(S) \equiv & -\left[\frac{1 - e^{-(r+\pi)T_1(S)}(1 + (r+\pi)T_1(S))}{(r+\pi)^2}\right] \frac{(2-\alpha)S - \alpha Q}{T_1(S)} \frac{s}{S} \\ & + \left[\frac{1 - e^{-(r+\pi)T_1(S)}}{r+\pi}\right](2-\alpha)s - \left[\frac{1 - e^{-rT_1(S)}}{r}\right]\alpha Q - \frac{c}{b} = 0, \end{aligned} \tag{40}$$

where

$$T_1(S) \equiv -\frac{\log(S/s)}{\pi}, \quad s = \frac{\alpha Q + rc/b}{2-\alpha}.$$

The theorem states that the objects above constitute a (Q,S,s) equilibrium if and only if $\varphi(Q) > 0$. The theorem also provides a characterization of the seller's present value of profits, $V(t)$, and of the seller's flow profit, $R(p)$. The proof of the theorem is nearly identical to the proof of Theorem 1 and is omitted.

Theorem 5. *A (Q,S,s) equilibrium exists iff $\varphi(Q) > 0$. If a (Q,S,s) equilibrium exists, it is unique and: (1) $V'(t) < 0$ for all $t \in (0, T_1)$ and $V'(t) = 0$ for all $t \in (T_1, T_1 + T_2)$; (2) $\hat{R}'(t) > 0$ for all $t \in (0, \hat{T})$, $\hat{R}'(t) < 0$ for all $t \in (\hat{T}, T_1)$ and $\hat{R}'(t) = 0$ for all $t \in (T_1, T_1 + T_2)$, where $\hat{R}(t) \equiv R(p(t))$ and $\hat{T} \in (0, T_1)$.*

It is immediate to verify that the condition for existence of a (Q,S,s) equilibrium is satisfied when c is small enough. It is also possible to show that an (S,s) equilibrium does not exist when c is small enough. The intuition behind these findings is the same as in the case of inflation. Because of search frictions, equilibrium requires prices to be sufficiently dispersed. However, when the menu cost is small, there would not be enough price dispersion if sellers were to follow an (S,s) rule as s and S would be too close to each other. Hence, when the menu cost is small, equilibrium requires sellers to switch to a (Q,S,s) rule which creates some additional price dispersion (and, indirectly, additional price stickiness). The main difference between the (Q,S,s) equilibrium with inflation and deflation is the origin of this extra price dispersion. With inflation, extra price dispersion is generated by sellers randomizing over the new price they choose after they pay the menu cost. With deflation, extra price dispersion is generated by sellers randomizing over the time at which they pay the menu cost.

The difference in the nature of the (Q,S,s) rule is reflected in the different features of equilibrium. With deflation, the present value of profits is decreasing as the real price drifts from s to S and it remains constant at its minimum as the price drifts from S to Q. With inflation, the present value of profits is first constant at its maximum and then strictly decreasing. With deflation, the flow profit rises, reaches its maximum and then falls as the price drifts from s to S, and it remains constant to its minimum as the real price drifts from S to Q. With inflation, the flow profit is at first constant, then it rises to its maximum and falls to its minimum. With deflation as with inflation, the density of the price distribution is log-uniform over $[s, S]$ and such that flow profits are constant over $[S, Q]$. Moreover, the density has a discontinuity at S which guarantees that the highest flow profit is attained in the $[s, S]$ interval. With deflation the discontinuity is caused by a fraction of sellers who pays the menu cost as soon as the real price reaches S. With inflation the discontinuity is caused by a fraction of sellers who reset the price to S after they pay the menu cost. With deflation, the density required to keep the flow profit constant in $[S, Q]$ is created by the rate at which sellers pay the menu cost. With inflation, it is created by the distribution of new prices.

Finally, let us explain why the (Q,S,s) pricing rule involves randomization over new prices with inflation and randomization over exit prices with deflation. In any region $[A, B]$ where sellers randomize, the density $F'(p)$ has to keep the flow profit constant. Note that the derivative of this density $F'(p)$ is lower than the derivative of the density $U'(p)$ of a log-uniform distribution with $U'(p) = F'(p)$. The density of a log-uniform distribution is important because it is the distribution that would emerge if sellers neither exited from nor directly entered in the interior of the interval $[A, B]$. With deflation, prices drift to the right. Hence, $F'(p) < U'(p)$ for all $p \in (A, B)$ given that U is a log-uniform distribution with $U'(A+) = F'(A+)$. This condition for $F'(p)$ can only be satisfied if some sellers exit as they drift from A to B. With inflation prices drift to the left. Hence, $F'(p) > U'(p)$ for all $p \in (A, B)$ given a log-uniform distribution with $U'(B-) = F'(B-)$ This condition for $F'(p)$ can only be satisfied if some sellers enter directly in the interior of (A, B).

5. INFLATION AND WELFARE

In this section, we study the effect of inflation on welfare when sellers follow a (Q,S,s) pricing rule. We find that inflation always leads to higher prices and lower welfare. This is in contrast with Bénabou (1988) and Diamond (1993) who study the welfare effect of inflation in models similar to ours except for the fact that sellers follow an (S,s) rule and who find that inflation tends to lower prices and a positive rate of inflation may maximize welfare.

We consider a version of the model with entry and exit of sellers. We assume that sellers can enter the market by paying the cost $C>0$. After entering the market, sellers set their nominal price, trade with buyers and occasionally pay the menu cost until they exit, an event which takes place at the rate $\delta>0$. A fixed measure of buyers enters the market in every interval of time of length dt. A fraction $\lambda(\tau)\alpha$ of buyers contacts 1 seller and a fraction $\lambda(\tau)(1-\alpha)$ contacts 2 sellers, where λ is a matching probability increasing in the seller–buyer ratio τ. The matching probability λ captures the idea that, if the seller–buyer ratio increases, it is easier for a buyer to find a seller. We denote as mdt the exogenous measure of buyers and as ndt the endogenous measure of sellers entering the market in an interval of time of length dt. We denote as $b(\tau)=\lambda(\tau)/\tau$ the ratio of buyers with at least one contact to sellers and assume b to be strictly decreasing in τ. We focus on the stationary equilibrium of the model as $\delta\to 0$, so that we can approximate the equilibrium objects with those derived in Section 3.

We measure welfare W as the sum of the lifetime utility of buyers and sellers entering the market during any interval of time of length dt, that is

$$W=m\lambda(\tau)dt[\alpha(Q-\mathbb{E}_1 p)+(1-\alpha)(Q-\mathbb{E}_2 p)]+ndt\left[V^*-C\right], \tag{41}$$

where $\mathbb{E}_1 p=\int pF'(p)dp$ and $\mathbb{E}_2 p=\int p2(1-F(p))F'(p)dp$. The first term in (41) is the expected lifetime utility of buyers entering the market. There are mdt buyers entering the market. With probability $\alpha\lambda(\tau)$, an entering buyer enjoys an expected utility of $Q-\mathbb{E}_1 p$, where $\mathbb{E}_1 p$ is the average price when the buyer contacts 1 seller. With probability $(1-\alpha)\lambda(\tau)$, an entering buyer enjoys an expected utility of $Q-\mathbb{E}_2 p$, where $\mathbb{E}_2 p$ is the average price when the buyer contacts two sellers. The second term in (41) is the expected lifetime utility of sellers entering the market. The measure of sellers entering the market is ndt. Each seller pays the cost C and then enjoys a lifetime utility of V^*. Note that, since we evaluate welfare at a stationary equilibrium, measuring welfare as the lifetime utility of entrants in the current period is the same as measuring welfare as the sum of lifetime utilities of entrants in the current and future periods.

It is clear from (41) that higher inflation increases (reduces) welfare if it leads to lower (higher) prices without affecting too much the buyer's matching probability $\lambda(\tau)$. In fact, if higher inflation leads to a decline in prices without lowering too much $\lambda(\tau)$, the lifetime utility of buyers increases. If higher inflation leads to an increase in prices without raising $\lambda(\tau)$ too much, the lifetime utility of buyers falls. In contrast, inflation does not affect the lifetime utility of sellers because free entry guarantees that $V^*=C$ and, hence, the second term in (41) is always equal to zero.

Now, let $(\tau_1,F_1,G_1,s_1,S_1,V_1^*)$ denote the (Q,S,s) equilibrium when the inflation rate is π_1 and let $(\tau_2,F_2,G_2,s_2,S_2,V_2^*)$ denote the (Q,S,s) equilibrium when the inflation rate is π_2, with $0<\pi_1<\pi_2<\overline{\pi}$. First, note that τ_i is the same in both equilibria. In fact, the free entry condition $V_i^*=C$ and the equilibrium seller's maximized profit $V_i^*=b(\tau_i)\alpha Q/r$ imply $\tau_i=rC/(\alpha Q)$ for $i=\{1,2\}$. Intuitively, since the inflation rate does not affect the seller's maximized profit, it does not affect the entry decision of sellers and, in turn, the seller–buyer ratio. Second, note that the lowest price s_i is the same in both equilibria. In fact, the optimality condition (20) implies $s_i=r(V^*/b(\tau_i)-c/b(\tau_i))/(2-\alpha)$ for $i=\{1,2\}$. Since the inflation rate affects neither V^* nor b, the lowest price is the same in both equilibria. Third, note that $S_1<S_2$. In fact, Theorem 2 establishes that, for a given b, S is strictly increasing in π.

We are now in the position to measure the effect of inflation on the price distribution F. For any $p\in[s,S_1]$, the difference $F_2(p)-F_1(p)$ is given by

$$F_2(p)-F_1(p)=\left[1-\frac{\alpha}{2(1-\alpha)}\frac{Q-S_2}{S_2}\right]\frac{\log(p/s)}{\log(S_2/s)}-\left[1-\frac{\alpha}{2(1-\alpha)}\frac{Q-S_1}{S_1}\right]\frac{\log(p/s)}{\log(S_1/s)}. \tag{42}$$

For any $p \in [S_1, S_2]$, the difference $F_2(p) - F_1(p)$ is given by

$$F_2(p) - F_1(p) = \left[1 - \frac{\alpha}{2(1-\alpha)} \frac{Q - S_2}{S_2}\right] \frac{\log(p/s)}{\log(S_2/s)} - \left[1 - \frac{\alpha}{2(1-\alpha)} \frac{Q-p}{p}\right]. \tag{43}$$

For any $p \in [S_2, Q]$, the difference $F_2(p) - F_1(p)$ is given by

$$F_2(p) - F_1(p) = 0. \tag{44}$$

It is easy to show that (42) is strictly negative for all $p \in (s, S_1]$. To see this, note that (42) can be written as the integral for S going from S_1 to S_2 of the derivative with respect to S of $F(p)$ in (16). This derivative has the same sign as $\alpha Q(1+\log(S/s)) - (1-\alpha)S$, which is strictly negative for any S in $[S_1, S_2]$. Similarly, it is easy to show that (43) is strictly negative for all $p \in [S_1, S_2)$. Everywhere else $F_2 - F_1$ is zero. Overall, F_2 first order stochastically dominates F_1. Therefore, the average prices $\mathbb{E}_1 p$ and $\mathbb{E}_1 p$ paid by buyers who sample respectively 1 and 2 sellers are strictly greater under F_2 than F_1. It then follows from (41) that welfare is strictly lower when the inflation is π_2 than π_1.

Our findings are summarized in the following theorem.

Theorem 6. *For any inflation rates $\pi_1, \pi_2 \in (0, \overline{\pi})$ with $\pi_1 < \pi_2$, the price distribution F is higher and welfare W is lower in the (Q,S,s) equilibrium associated with π_2 than in the (Q,S,s) equilibrium associated with π_1.*

The intuition behind Theorem 6 is simple. In a (Q,S,s) equilibrium, inflation has no effect on the sellers' maximum profit for a given seller–buyer ratio. This observation implies that inflation has no effect on τ and, in turn, on the buyers' matching probability $\lambda(\tau)$. The observation that inflation has no effect on the sellers' maximum profit also implies that, while higher inflation increases the resources spent by sellers to change their price, the increase in expenditures is passed by the sellers to the buyers through higher prices. Overall, higher inflation only raises prices and, hence, lowers welfare. Theorem 6 stands in contrast with Bénabou (1988) and Diamond (1993) who show that welfare may be maximized at a strictly positive inflation rate in models where all buyers are captive and, hence, sellers follow a standard (S,s) rule. Indeed, in an (S,s) equilibrium, higher inflation typically lowers the sellers' maximum profit. For this reason, the seller–buyer ratio τ and, in turn, the buyers' matching probability $\lambda(\tau)$ tends to fall. For the same reason, sellers pay the menu cost at a lower s and, in turn, the stationary price distribution falls. As long as the decline in τ does not affect $\lambda(\tau)$ too much, inflation increases welfare. If $\lambda(\tau)$ is a concave function, welfare is typically hump-shaped in inflation.

In our model, equilibrium is such that sellers follow a (Q,S,s) rule when inflation is relatively low and an (S,s) rule when inflation is relatively high. In the (Q,S,s) region, welfare is strictly decreasing in inflation. In the (S,s) region, welfare is typically hump-shaped in inflation. Therefore, the welfare-maximizing inflation rate may either be zero or somewhere in the (S,s) region. However, for the parametrizations of the model presented in the next section and for a standard telephone-line matching function,[15] we find that welfare is maximized at an inflation rate of zero.

15. The telephone-line matching function is such that $\lambda(\tau) = \tau/(1+\tau)$ and $b(\tau) = 1/(1+\tau)$.

Theorem 6 generalizes to the version of the model with a divisible good. For this version of the model, welfare is given by

$$W = m\lambda(\tau)dt\Big[\alpha\Big(Q^{\theta}-\mathbb{E}_1 p^{\theta}\Big)/\theta+(1-\alpha)(Q^{\theta}-\mathbb{E}_2 p^{\theta})/\theta\Big]+ndt\big[V^{*}-C\big]. \tag{45}$$

Just as in the case of an indivisible good, the only effect of inflation in a (Q,S,s) equilibrium is to increase the stationary distribution of prices F. Therefore, an increase in inflation lowers the lifetime utility of entering buyers (the first term in (45)), leaves unchanged the lifetime utility of entering sellers net of the entry cost (the second term in (45)) and lowers overall welfare. Notice that, in contrast to the case of an indivisible good, higher prices lower the buyers' lifetime utility not only because buyers pay higher prices for the good but also because buyers purchase fewer units of the good.

6. QUANTITATIVE ANALYSIS

The theoretical analysis revealed two key differences between a (Q,S,s) equilibrium and a standard (S,s) equilibrium. First, in a (Q,S,s) equilibrium, both search frictions and menu costs contribute to the stickiness of nominal prices, while in an (S,s) equilibrium only menu costs do. Second, in a (Q,S,s) equilibrium, inflation increases prices and lowers welfare, while in an (S,s) equilibrium this need not to be the case. These findings suggest that the effect of an unanticipated monetary injection and the effect of an increase in the growth rate of money depend on whether sellers follow a (Q,S,s) or an (S,s) rule. In this section, we bring our model to the data to find out what type of equilibrium prevails and to measure the relative contribution of search frictions and menu costs to price stickiness. The quantitative analysis in this section is exploratory, as our model is too stark to capture all features of the data. Nevertheless, we believe that our findings are still informative as back-of-the-envelope calculations.

The parameters to calibrate are the real interest rate, r, the inflation rate, π, the arrival rate of buyers, b, the fraction of captive buyers, α, the buyer's valuation of the good, Q, the seller's menu cost, c, and the seller's cost of producing the good, k (which we had previously set to zero to keep the algebra simpler). We set the real interest rate r to 5% and the inflation rate π to 3%.[16] We note that the equilibrium objects S and s increase by a factor λ and the location of the equilibrium distributions F and G shifts by a factor λ whenever the parameters Q, k and c increase by λ.[17] Hence, we can normalize the buyer's valuation Q to 1. Further, we note that the equilibrium objects F, G, S and s depend on c and b only though their ratio c/b.[18] Hence, we can normalize the inflow of buyers b to 1 and interpret the seller's menu cost as a fraction of b. We set k to be equal to 60% of Q, a value which generates markups around 10% in our benchmark

16. We choose 5% for r, as this is a common choice for the real interest rate. We choose 3% for π, as this is close to the aggregate rate of inflation over the period 1998–2014, from which the data on the duration of prices and on the dispersion of prices is collected. Using alternative targets for r and π does not significantly affect our findings.

17. Formally, the following homogeneity property holds: Let (F,G,s,S,V^*) be a (Q,S,s) equilibrium given the parameters Q, k and c. Then, for all $\lambda>0$, $(\hat{F},\hat{G},\hat{s},\hat{S},\hat{V}^*)$ is a (Q,S,s) equilibrium given the parameters λQ, λk and λc, where $\hat{F}(\lambda p)=F(p)$, $G(\lambda p)=G(p)$, $\hat{s}=\lambda s$, $\hat{S}=\lambda S$, and $\hat{V}^*=\lambda V^*$. An analogous property holds for an (S,s) equilibrium.

18. Formally, the following homogeneity property holds: Let (F,G,s,S,V^*) be a (Q,S,s) equilibrium given the parameters b and c. Then, (F,G,s,S,V^*) is a (Q,S,s) equilibrium given the parameters λb and λc, for all $\lambda>0$. An analogous property holds for an (S,s) equilibrium. The intuition for these properties is straightforward. The seller's objective function is an integral of flow revenues that are premultiplied by b and an occasional menu cost payment c. Dividing the objective function by b, we obtain a new objective function which depends only on c/b. Clearly, the new objective function has the same solution as the original one.

calibration.[19] Finally, we calibrate α and c to match empirical measures of price dispersion and price stickiness.

Nakamura and Steinsson (2008) measure price stickiness for consumer goods using the Bureau of Labor Statistics microdata underlying the Consumer Price Index. During the 1998–2005 period, they find that the average duration of nominal prices is 7.7 months if sales and product substitutions are included in the data, and 13 months if sales and substitutions are excluded. Kaplan *et al*. (2016) measure price dispersion for consumer goods using the Kielts-Nielsen Homescan Dataset. They measure the standard deviation of prices for the same item (defined by its Unique Product Code) in the same market (defined as a Scantrack Metro Area) in the same week. They find that the average standard deviation of prices is 15%. They break down the price of each good at each store into a store component and a store-good component, and estimate an ARMA process for each of the two components. The estimation reveals that the variance of the store component accounts for 15.5% of the overall variance of the price, while the variance of the store-good component accounts for the remaining 84.5%. The estimation also reveals that 36% of the variance of the store-good component is due to persistent differences and 64% is due to temporary differences in the store-good component of prices, the latter presumably reflecting sales.

Since there are no temporary sales and product substitutions in our model, it is natural to target measures of average price duration computed excluding sales and substitutions. Since our model abstracts from sellers' heterogeneity and temporary sales, it is natural to target the fraction of price dispersion caused by persistent differences in the store-good component of prices, and exclude the fraction of price dispersion caused by differences in the store component of prices (which presumably are due to differences among sellers) and by temporary differences in the store-good component of prices (which presumably are due to temporary sales). Therefore, the baseline calibration targets are 13 months for price duration and 8.2% for the standard deviation of log prices.[20] Since different products feature very different price stickiness and price dispersion (see, *e.g*., Nakamura and Steinsson, 2008 and Kaplan and Menzio, 2015), it is natural to consider some alternative calibration targets.

Before turning to the outcome of the calibration, it is useful to explain why price stickiness and price dispersion identify both search frictions (*i.e*. the parameter α) and menu costs (*i.e*. the parameter c). First, consider a pure menu cost version of the model—*i.e*. a version of the model in which, since $\alpha = 1$, sellers follow an (S,s) rule and the menu cost is the only cause of price stickiness. In a pure menu cost model, the distance between s and Q uniquely pins down the distribution of prices and, in turn, the standard deviation of log prices across sellers (*i.e*. price

19. The empirical literature on markups has not reached a consensus on the magnitude of the ratio between price and marginal cost. Basu and Fernald (1997) find gross markups between 0.66 and 1.32 depending on the sector and use of instrumental variables. They conclude that the typical industry has small markups over marginal cost. Klette (1999) reaches a similar conclusion using a different estimation technique. De Loecker and Warzynski (2012) find gross markups between 1.03 and 1.22 depending on the estimation strategy. In our benchmark calibration, the markup is 10%. When we target a higher (lower) markup, the calibrated value of the menu cost and its contribution to price stickiness falls (increases).

20. Footnote 5: The standard deviation of the price of the same good across different sellers in the same market and in the same week is 15%, which implies a variance of 2.25%. The fraction of the variance that is due to variation in the store component of prices is 15.5%. The fraction of the variance that is due to variation in the transitory part of the store-good component of prices is 64% of the remaining 84.5%. The variance that is due to variation in the persistent part of the store-good component of prices—which we argued is the proper target for our model—is 36% of 84.5% of 2.25%. This is approximately equal to a variance of 0.68% or, equivalently, a standard deviation of 8.2%.

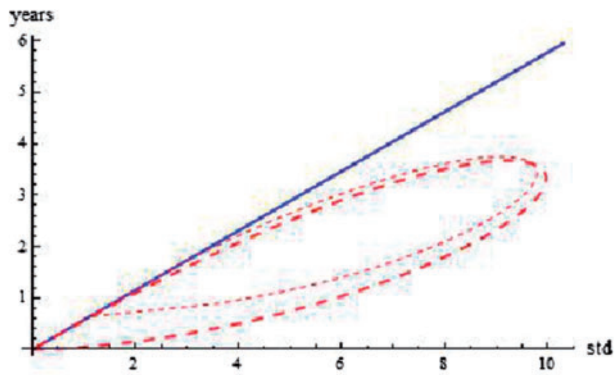

FIGURE 3

Price dispersion and price stickiness.

Notes: Standard deviation of log prices and average duration of prices in years. Solid curve: pure menu cost model ($\alpha = 1$). Long-dashed curve: pure search model ($c \to 0$). Short-dashed curve: search model with menu cost of 0.01. All models computed with $\pi = 3\%$ and $r = 5\%$. Search models computed with $k = 0.6$

dispersion) and the average duration of the price of an individual seller (*i.e.*, price stickiness).[21] For $s = Q$, price dispersion and price stickiness are both zero. As we lower s, dispersion and stickiness both increase. The solid curve at the top of Figure 3 is the locus of price dispersion and price stickiness generated by the menu cost model as we go from $s = Q$ to $s \to 0$.

Next, consider a pure search friction version of the model—that is, a version of the model in which c is arbitrarily small and, hence, search frictions are the only cause of price stickiness. In a pure search friction model, the fraction α of captive buyers is the unique determinant of both price dispersion and price stickiness. For $\alpha = 0$, the equilibrium price distribution is degenerate at the competitive price and, hence, dispersion and stickiness are both zero. As we increase α, the equilibrium price distribution starts to spread out and, hence, dispersion and stickiness both increase. As α converges to 1, the equilibrium price distribution converges to the monopoly price and, hence, dispersion and stickiness converge back to zero. The long-dashed curve at the bottom of Figure 3 is the locus of price dispersion and price stickiness generated by the pure search model as we go from $\alpha = 0$ to $\alpha = 1$.

In general, the pure menu cost model and the pure search friction model cannot simultaneously match the targets for price dispersion and price stickiness. To match these targets, we generally need a particular combination of menu costs and search frictions. By increasing the menu cost c, the locus of combinations of price dispersion and price stickiness generated by the search friction model shifts monotonically up towards the solid blue curve (see the short-dashed curve line in Figure 3). By finding the manifold that passes through the targets for price dispersion and price stickiness, we identify the menu cost c. By finding the point on the manifold that matches the targeted combination of price dispersion and price stickiness, we identify the fraction of captive buyers α.

To understand the results our calibration, it is sufficient to look at Figure 3 again. When the pure menu cost model is asked to match the average price dispersion observed in the data (which net of the store component of prices and of temporary sales is 8.2%), it generates a price

21. This statement applies to all models in which sellers follow an (S,s) rule, as in all of these models the distance between the upper and the lower bound of the distribution is the sole determinant, together with inflation, of price dispersion and price stickiness.

TABLE 1
Calibration targets and outcomes, basic model

Calibration targets	(1)	(2)	(3)	(4)
Standard deviation of prices	8.2%	8.2%	5.5%	5.5%
Average duration of prices	13 mo.	24 mo.	13 mo.	24 mo.
Calibration outcomes	(1)	(2)	(3)	(4)
Captive buyers α	7.9%	8.5%	2.9%	2.9%
Menu cost c	0	0.12%	0.16%	5.6%
Contribution of c to stickiness	(1)	(2)	(3)	(4)
Counterfactual with $c=0$	0%	2%	7%	50%
Counterfactual with $\alpha=1$	0%	8%	15%	50%

duration of almost 5 years, while price duration in the data is on average 13 months. In other words, the pure menu cost model generates four times more price stickiness than in the data when required to match the empirical price dispersion. In contrast, when the pure search model is asked to match the empirical dispersion of prices, it generates a price duration of approximately 1.5 years. That is, a model with search frictions alone can produce roughly the right amount of price stickiness when required to match the empirical price dispersion. For some choices of calibration targets, the pure search model generates a little more stickiness than in the data. In these cases, the best fit of the model to the data is achieved by setting the menu cost to zero. For other choices of calibration targets, the pure search model generates a little less stickiness than in the data. In these cases, the best fit of the model to the data is achieved with a small, positive menu cost.

In Table 1, we report the outcomes of the calibration for four different combinations of targets. In column 1, we target the average standard deviation of prices of 8.2% and the average price duration of 13 months. Given these targets, the estimated fraction of captive buyers is 8%, the estimated menu cost is 0, and the equilibrium is such that sellers follow a (Q,S,s) rule with $s=S$ as in Head *et al*. (2012). If we target a below-average standard deviation of prices and/or an above-average duration of prices, the estimated menu cost is positive. If we target the average price dispersion of 8.2% and an above-average price duration of 24 months (column 2), the estimated menu cost is 0.12%, which is equal to 3.5% of the annuitized present value of seller's profits. If we target a below-average price dispersion of 5.5% and the average price duration of 13 months (column 3), the estimated menu cost is 0.16%. If we target a below-average price dispersion of 5.5% and an above-average price duration of 24 months (column 4), the estimated menu cost is 5.6%. In all these calibrations, the equilibrium is such that sellers follow a (Q,S,s) rule. Therefore, an increase in inflation is—at least locally—welfare decreasing.

Next, we measure the contribution of menu costs and search frictions to price stickiness. We consider two alternative metrics. The first metric is the complement to 1 of the ratio between the counterfactual average duration of prices when the menu cost c is set to 0 and the actual duration of prices. The second metric is the ratio between the counterfactual average duration of prices when the fraction of captive buyers α is set to 1 and the actual average duration of prices. This is a measure of the contribution of menu costs to price stickiness because search frictions do not create any price stickiness when $\alpha=1$. For both metrics, the contribution of search frictions to price stickiness can be recovered as the fraction that is not accounted for by menu costs.

In the benchmark calibration, the estimated menu costs are zero and so is their contribution to price stickiness. In the high-duration calibration (column 2), the estimated menu costs are positive and their contribution to price stickiness is equal to 2% when measured as the complement to 1 of the ratio between price stickiness with $c=0$ and actual price stickiness and it is equal to 8% when measured as the ratio between price stickiness with $\alpha=1$ and actual price stickiness. In the low-dispersion calibration (column 3), the two measures of the contribution of menu costs to price stickiness are 7% and 15%, respectively. In the low-dispersion, high-duration calibration (column 4), the two measures of the contribution of menu costs to price stickiness are both equal to 50%.

The main take-away of Table 1 is that both search frictions and menu costs contribute to price stickiness, although search frictions appear to be relatively more important. This suggests that theories of price stickiness that abstract from search frictions—for example Dotsey *et al*. (1999)—are likely to overestimate the magnitude of menu costs and, in turn, the importance of nominal price rigidity in the transmission of monetary policy shocks to the real side of the economy. Similarly, theories of price stickiness that abstract from menu costs—for example Head *et al*. (2012)—are likely to underestimate the importance of nominal price rigidities as a channel of transmission of monetary policy.

Our model is very simple and the quantitative findings in this section should be taken with a grain of salt. Indeed, our model makes two obviously counterfactual predictions. First, the model predicts a price distribution with a decreasing density, while in the data the typical price distribution is hump-shaped (see, *e.g*., Kaplan and Menzio, 2015). The discrepancy is somewhat superficial as the model is meant to capture only the dispersion in the store-good component of prices and not the overall dispersion of prices. Second, the model predicts a distribution of price changes with two modes at $(S-s)/s$ and $(Q-s)/s$ and, most importantly, no negative price changes. In the data, the distribution of price changes is hump-shaped and 30% of price changes are negative (see, *e.g*., Klenow and Kryvtsov, 2008 or Alvarez *et al*., 2016). The fact 30% of price changes are negative suggests that sellers change their price for reasons other than keeping up with inflation. And if sellers have reasons to change their price that are not included in the model, our calibration will underestimate the magnitude of the menu cost and its contribution to price stickiness.

There are two approaches to address the issue of negative price changes. The first approach is to develop and calibrate a version of the model in which sellers face idiosyncratic shocks that sometime induce them to reduce their nominal price. In an earlier version of the article, we pursued this approach and found a larger contribution of menu costs to price stickiness (albeit often still smaller than the contribution of search frictions). The second approach is to calibrate the model to a high-inflation economy where the predominant reason for changing prices is keeping up with the aggregate price level and the fraction of negative price changes is typically quite small. We pursue this approach by calibrating the model to Israel during the period 1978–1979. Lach and Tsiddon (1992) document that in this period inflation in Israel was 3.9% per month, average price duration was 2.2 months and price dispersion was 16%. Most importantly, they show that the fraction of negative price changes was only 5%. We calibrate the model to the inflation, price stickiness and price dispersion reported by Lach and Tsiddon (1992) under the additional assumptions that the fraction of price dispersion caused by differences in the store-good component of prices was the same as in the U.S. and that the markup was 10%. We find that the contribution of the menu cost to price stickiness is larger than in the baseline calibration for the U.S., but still smaller than the contribution of search frictions. Specifically, the contribution of the menu cost to the observed duration of prices is 43% if measured using the $c=0$ counterfactual or 40% if measured using the $\alpha=1$ counterfactual.

7. CONCLUSIONS

The paper studied the properties of equilibrium in a version of the search-theoretic model of price dispersion of Burdett and Judd (1983) in which sellers face a menu cost to adjust their nominal price. We showed that, when the menu cost is small, equilibrium is such that sellers follow a (Q,S,s) pricing rule. According to this rule, a seller lets inflation erode the real value of its nominal price until it reaches a point s, then pays the menu cost and resets the nominal price so that its real value is a random draw from a distribution with support $[S,Q]$. When the menu cost is large, equilibrium is such that sellers follow a standard (S,s) rule. In a (Q,S,s) equilibrium both search frictions and menu costs contribute to price stickiness, while in an (S,s) equilibrium only menu costs do. In a (Q,S,s) equilibrium inflation always increases prices and lowers welfare, while in an (S,s) equilibrium this is typically not the case. In all our calibrations, the equilibrium involves (Q,S,s) pricing and search frictions are the main source of nominal price stickiness. These findings may be important to further our understanding of the effects of monetary policy on the real side of the economy and on welfare.

APPENDIX

A. PROOF OF THEOREM 1

(i) Part (i) of Theorem 1 states that a (Q,S,s) equilibrium exists if and only if $\varphi(Q)>0$. The function $\varphi(S)$ is defined as

$$\begin{aligned}\varphi(S)\equiv&\left[\frac{1-e^{-(r+\pi)T_2(S)}(1+(r+\pi)T_2(S))}{(r+\pi)^2}\right]\frac{(2-\alpha)S-\alpha Q}{T_2(S)}\\&+\left[\frac{1-e^{-(r+\pi)T_2(S)}}{r+\pi}\right]\alpha Q+e^{-rT_2(S)}\left(\frac{\alpha Q}{r}-\frac{c}{b}\right)-\frac{\alpha Q}{r}=0,\end{aligned}\tag{A1}$$

where

$$T_2(S)\equiv\frac{\log(S/s)}{\pi},\quad s=\frac{\alpha Q-rc/b}{2-\alpha}.\tag{A2}$$

It is straightforward to verify that $\varphi(S)$ has the following properties: (1) $\varphi(S)<0$ for all $S\in[s,\alpha Q/(2-\alpha)]$; (2) $\varphi'(S)>0$ for all $S\in[s,Q]$.

Suppose that $\varphi(Q)\le 0$. Then a (Q,S,s) equilibrium does not exist. On the way to a contradiction, let (F,G,S,s,V) be a (Q,S,s) equilibrium. As proved in Section 3, if (F,G,S,s,V) is a (Q,S,s) equilibrium, then S must be a solution to the equation $\varphi(S)=0$ and it must belong to the interval (s,S). However, $\varphi(S)=0$ cannot admit any solution in the interval (s,Q) because $\varphi(S)<0$ for all $S\in(s,Q)$.

Conversely, suppose that $\varphi(Q)>0$. Then, the equation $\varphi(S)=0$ admits at most one solution in the interval (s,Q) because $\varphi(s)<0$ and $\varphi(Q)\le 0$ and $\varphi(S)$ is strictly increasing in S. Let S^* denote this solution. Let s^* be defined as in equation (20) for $S=S^*$. Let F^* be defined as in equations (15) and (16) for $S=S^*$ and $s=s^*$. Let G^* be defined as in equations (17)–(19) for $S=S^*$ and $s=s^*$. Let V^* be defined as in equation (14). Moreover, let T_1^* and T_2^* be defined, respectively, as $\log(Q/S^*)/\pi$ and $\log(S^*/s^*)/\pi$ and let $R(p)$ be defined as in (2) for $F=F^*$. To prove that the tuple (F^*,G^*,S^*,s^*,V^*) constitutes a (Q,S,s) equilibrium, we need to verify that it jointly satisfies the optimality conditions (3), (4), (5), (7), and (8) and the stationarity conditions (11), (12), and (13). In addition, we need to verify that $S^*\in(s^*,Q)$, $s^*\in[0,Q)$, F^* is a CDF with support $[s^*,Q]$ and G^* is a CDF with support $[S^*,Q]$.

The tuple (F^*,G^*,S^*,s^*,V^*) satisfies the stationarity condition (11) because, for all $p\in(s^*,S^*)$, we have

$$\begin{aligned}F^{*\prime}(p)p&\equiv\left[1-\frac{\alpha}{2(1-\alpha)}\frac{Q-S^*}{S^*}\right]\frac{1}{\log(S^*/s^*)}\\&=F^{*\prime}(s^*)s^*.\end{aligned}\tag{A3}$$

Similarly, the stationarity condition (12) is satisfied because, for all $p \in (S^*, Q)$, we have

$$\begin{aligned} F^{*\prime}(p)p &\equiv \frac{\alpha}{2(1-\alpha)}\frac{Q}{p} \\ &= \left[1-\frac{\alpha}{2(1-\alpha)}\frac{Q-S^*}{S^*}\right]^{-1}\left[1-\frac{\alpha}{2(1-\alpha)}\frac{Q-S^*}{S^*}\right]\frac{s^*}{\log(S^*/s^*)}\frac{\alpha\log(S^*/s^*)Q}{2(1-\alpha)s^*p} \\ &= F^{*\prime}(s)(1-G^*(p))s^*. \end{aligned} \tag{A4}$$

Moreover, notice that F^* is a continuous CDF with support $[s^*, Q]$. In fact, $F^*(s^*)=0$, $F^*(Q)=1$ and $F^{*\prime}(p)>0$ for all $p \in [s^*, Q]$. Since $F^*(S-)=F^*(S+)$, the stationarity condition (13) is also satisfied.

The tuple $(F^*, G^*, S^*, s^*, V^*)$ satisfies the optimality condition (3) because

$$\begin{aligned} r(V^*-c) &= b\alpha Q - cr \\ &= b[\alpha+2(1-\alpha)(1-F^*(s^*))]s^* \\ &= R(s^*). \end{aligned} \tag{A5}$$

The optimality condition (7) is satisfied because, for all $t \in [0, T_1^*]$, we have

$$\begin{aligned} R(Qe^{-\pi t}) &= b\big[\alpha+2(1-\alpha)(1-F^*(Qe^{-\pi t}))\big]Qe^{-\pi t} \\ &= b\alpha Q = rV^*. \end{aligned} \tag{A6}$$

The optimality condition (8) is satisfied because $\varphi(S^*)=0$ implies

$$\int_{T_1^*}^{T_1^*+T_2^*} e^{-r(x-T_1^*)}R(Qe^{-\pi x})dx + e^{-rT_2^*}\big(V^*-c\big) - V^* = 0. \tag{A7}$$

Moreover, $S^* \in (s^*, Q)$ and the assumption $c \in (0, b\alpha Q/r)$ implies

$$s^* \equiv \frac{\alpha Q - rc/b}{2-\alpha} \in (0, Q). \tag{A8}$$

The optimality condition (7) together with the optimality condition (8) guarantees that the seller's value $V(t)$ is equal to V^* for all $t \in [0, T_1^*]$. Now, we need to verify that the optimality condition (6) That is, we need to verify that the seller's value $V(t)$ is non-greater than V^* for all $t \in [T_1^*, T_1^*+T_2^*]$ To this aim, notice that $V(t)$ satisfies the differential equation

$$\begin{aligned} &rV(t) = \hat{R}(t) + V'(t), \\ &\hat{R}(t) \equiv R(Qe^{-\pi t}). \end{aligned} \tag{A9}$$

For $t \in [T_1^*, T_1^*+T_2^*]$, $\hat{R}(t)$ is given by

$$\hat{R}(t) = e^{-\pi(t-T_1^*)}b\left\{(2-\alpha)S^* - \big[(2-\alpha)S^* - \alpha Q\big]\left[1-\frac{\pi(t-T_1^*)}{\log(S^*/s^*)}\right]\right\}. \tag{A10}$$

The derivative of $\hat{R}(t)$ with respect to t is given by

$$\hat{R}'(t) = \pi e^{-\pi(t-T_1^*)}b\left\{\big[(2-\alpha)S^* - \alpha Q\big]\left[1+\frac{1-\pi(t-T_1^*)}{\log(S^*/s^*)}\right] - (2-\alpha)S^*\right\}. \tag{A11}$$

Notice that $\hat{R}'(t)$ has the same sign as

$$\sigma(t) = \big[(2-\alpha)S^* - \alpha Q\big]\left[1+\frac{1-\pi(t-T_1^*)}{\log(S^*/s^*)}\right] - (2-\alpha)S^*. \tag{A12}$$

It is straightforward to verify that $\sigma(t)$ is strictly decreasing in t for all $t \in [T_1^*, T_1^*+T_2^*]$ and that $\sigma(T_1^*)$ is strictly positive (*negative*) if $(2-\alpha)S^*$ is strictly greater (*smaller*) than $\alpha Q(1+\log(S^*/s^*))$. Therefore, if $(2-\alpha)S^* > \alpha Q(1+\log(S^*/s^*))$, $\hat{R}(t)$ is first strictly increasing and then strictly decreasing over the interval $[T_1^*, T_1^*+T_2^*]$. If $(2-\alpha)S^* < \alpha Q(1+\log(S^*/s^*))$, $\hat{R}(t)$ is strictly decreasing for all $t \in [T_1^*, T_1^*+T_2^*]$. Notice that $\hat{R}(t)$ cannot be increasing for all $t \in [T_1^*, T_1^*+T_2^*]$ because $\hat{R}(T_1^*) = b\alpha Q$ and $\hat{R}(T_1^*+T_2^*) = b\alpha Q - cr$.

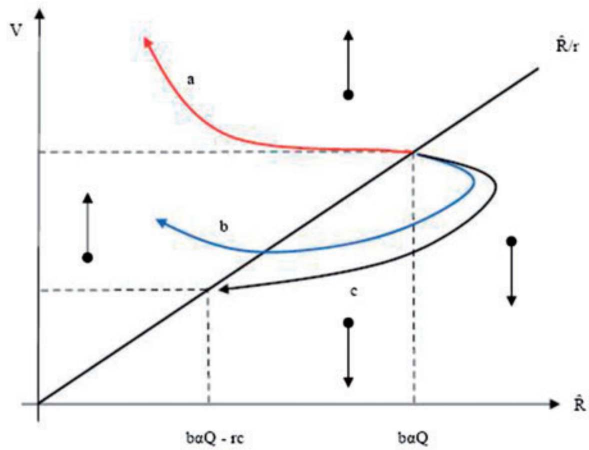

FIGURE 4
Joint Dynamics of $V(t)$ and $\hat{R}(t)$

Consider the phase diagram in Figure 4, which describes the differential equation (A9). The line passing through the origin denotes the locus of points $(\hat{R},V)$ such that $V=\hat{R}/r$ and, hence, $V'=0$. Any point below the line is such that $V<\hat{R}/r$ and, hence, $V'<0$. Any point above the line is such that $V>\hat{R}/r$ and, hence, $V'>0$. From (7) and (8), it follows that $\hat{R}(T_1^*)=b\alpha Q$ and $V(T_1^*)=b\alpha Q/r$. Hence, the point $(\hat{R}(T_1^*),V(T_1^*))$ lies on the line and $V'(T_1^*)=0$. From (14) and (20), it follows that $\hat{R}(T_1^*+T_2^*)=b\alpha Q-cr$ and $V(T_1^*+T_2^*)=b\alpha Q/r-c$. Hence, the point $(\hat{R}(T_1^*+T_2^*),V(T_1^*+T_2^*))$ lies on the line and $V'(T_1^*+T_2^*)=0$.

Now, we want to find the trajectory that the pair $(\hat{R}(t),V(t))$ follows as it goes from the point $(b\alpha Q,b\alpha Q/r)$ at $t=T_1^*$ to the point $(b\alpha Q-cr,b\alpha Q/r-c)$ at $t=T_1^*+T_2^*$. Notice that $(2-\alpha)S^*$ must be greater than $\alpha Q(1+\log(S^*/s^*))$. In fact, if $(2-\alpha)S^*\leq\alpha Q(1+\log(S^*/s^*))$, $\hat{R}(t)$ is strictly decreasing for all $t\in[T_1^*,T_1^*+T_2^*]$. As illustrated by the trajectory (a) in the phase diagram, this implies that $(\hat{R}(t),V(t))$ exits the initial point $(b\alpha Q,b\alpha Q/r)$ from the left, enters the region where $V'(t)>0$, and remains in that region for all $t\in(T_1^*,T_1^*+T_2^*)$. Thus, $V(T_1^*+T_2^*)>V(T_1^*)=b\alpha Q/r$, which contradicts the fact that $V(T_1^*+T_2^*)=b\alpha Q/r-c$.

Since $(2-\alpha)S^*>\alpha Q(1+\log(S^*/s^*))$, $\hat{R}(t)$ is first increasing and then decreasing over the interval $[T_1^*,T_1^*+T_2^*]$. As illustrated by trajectories (b) and (c) in the phase diagram, this implies that $(\hat{R}(t),V(t))$ exits the initial point $(b\alpha Q,b\alpha Q/r)$ from the right, enters the region where $V'(t)<0$, and remains in that region until it crosses the black line either at some $\hat{t}<T_1^*+T_2^*$, as in trajectory (b), or at $T_1^*+T_2^*$, as in trajectory (c). If $(\hat{R}(t),V(t))$ crosses the line at some $\hat{t}<T_1^*+T_2^*$, then $\hat{R}'(\hat{t})<0$. This implies that $V'(t)>0$ and $\hat{R}'(t)<0$ for all $t\in(\hat{t},T_1^*+T_2^*)$. Thus, $(\hat{R}(t),V(t))$ cannot reach the end point $(b\alpha Q-cr,b\alpha Q/r-c)$. Therefore, the pair $(\hat{R}(t),V(t))$ must follow the trajectory (c). Along this trajectory, $V(t)$ is strictly decreasing. Hence, the optimality condition (6) is satisfied. Moreover, along this trajectory, $\hat{R}(t)\geq r(V^*-c)$. Hence, the optimality condition (4) is also satisfied.

To complete the existence proof, we still need to verify that G^* is a CDF with support $[S^*,Q]$. To this aim, recall that G^* is given by

$$G^*(p)\equiv\begin{cases}0, & \text{if } p<S^*,\\ 1-\left[1-\dfrac{\alpha}{2(1-\alpha)}\dfrac{Q-S^*}{S^*}\right]^{-1}\dfrac{\alpha\log(S^*/s^*)Q}{2(1-\alpha)p}, & \text{if } p\in[S^*,Q),\\ 1, & \text{if } p\geq Q.\end{cases} \tag{A13}$$

The distribution function $G^*(p)$ has the following properties: (1) $G^*(p)=0$ for all $p<S^*$; (2) $G^*(S^*)\geq 0$ if and only if $(2-\alpha)S^*\geq\alpha Q[1+\log(S^*/s^*)]$; (3) $G^{*\prime}(p)>0$ for all $p\in(S^*,Q)$ if and only if $S^*>\alpha Q/(2-\alpha)$; (4) $G^*(Q-)\leq 1$ if and only if $S^*>\alpha Q/(2-\alpha)$; (5) $G^*(p)=1$ for all $p\geq Q$. Therefore, $G^*(p)$ is a proper CDF with support $[S^*,Q]$ if and only if $S^*>\alpha Q/(2-\alpha)$ and $(2-\alpha)S^*\geq\alpha Q[1+\log(S^*/s^*)]$. We have already established that both conditions hold.
(ii) Part (ii) of Theorem 1 states that: (1) $V'(t)=0$ for all $t\in(0,T_1^*)$ and $V'(t)<0$ for all $t\in(T_1^*,T_1^*+T_2^*)$; (2) $\hat{R}'(t)=0$ for all $t\in(0,T_1^*)$, $\hat{R}'(t)>0$ for all $t\in(T_1^*,\hat{T})$ and $\hat{R}'(t)<0$ for all $t\in(\hat{T},T_1^*+T_2^*)$. Both properties have been established while proving part (i). ‖

B. PROOF OF THEOREM 2

(i) In the proof of Theorem 1, we showed that a (Q,S,s) equilibrium exists if and only if the equation $\varphi(S,c)=0$ admits a solution for $S\in(\alpha Q/(2-\alpha),Q)$. The function $\varphi(S,c)$ is given by

$$\varphi(S,c)=\left[\frac{1-e^{-(r+\pi)T_2(S,c)}(1+(r+\pi)T_2(S,c))}{(r+\pi)^2T_2(S,c)}\right][(2-\alpha)S-\alpha Q] \\ +\left[\frac{1-e^{-(r+\pi)T_2(S,c)}}{r+\pi}\right]\alpha Q+e^{-rT_2(S,c)}\left(\frac{\alpha Q}{r}-\frac{c}{b}\right)-\frac{aQ}{r}, \tag{B1}$$

where $T_2(S,c)$ is given by

$$T_2(S,c)=\frac{\log(S/s(c))}{\pi},\quad s(c)=\frac{\alpha Q-rc/b}{2-\alpha}. \tag{B2}$$

Let $\varphi_S(S,c)$ and $\varphi_c(S,c)$ denote the derivatives of $\varphi(S,c)$ with respect to S and c. The derivative $\varphi_S(S,c)$ is given by

$$\varphi_S(S,c)=\left[\frac{1-e^{-(r+\pi)T_2(S,c)}(1+(r+\pi)T_2(S,c))}{(r+\pi)^2}\right]\frac{(2-\alpha)[\log(S/s(c))-1]+\alpha Q}{\log(S/s(c))^2}. \tag{B1}$$

The derivative $\varphi_c(S,c)$ is given by

$$\varphi_c(S,c)=-\frac{e^{-rT_2(S,c)}}{b}-\left[\frac{1-e^{-(r+\pi)T_2(S,c)}(1+(r+\pi)T_2(S,c))}{(r+\pi)^2}\right]\left[\frac{(2-\alpha)S-\alpha Q}{b\log(S/s(c))^2}\right]\frac{\pi r}{(2-\alpha)s(c)}. \tag{B2}$$

It is straightforward to verify that $\varphi_S(S,c)$ is strictly positive and $\varphi_c(S,c)$ is strictly negative for all $S\in[\alpha Q/(2-\alpha),Q]$.

Let $S^*(c)$ denote the solution to the equation $\varphi(S,c)=0$ with respect to S. For $c=0$, $S^*(c)$ is equal to $\alpha Q/(2-\alpha)$. Since $\varphi_S(S,c)$ is strictly positive and $\varphi_c(S,c)$ is strictly negative for all $S\in[\alpha Q/(2-\alpha),Q]$, it follows that $S^*(c)$ is strictly greater than $\alpha Q/(2-\alpha)$ and strictly increasing for all $c>0$. Moreover, there exists a $\overline{c}>0$ such that $S^*(\overline{c})=Q$. It then follows that, for all $c\in(0,\overline{c})$, $S^*(c)\in(\alpha Q/(2-\alpha),Q)$ and a (Q,S,s) equilibrium exists. In contrast, for $c\geq\overline{c}$, $S^*(c)\geq Q$ and a (Q,S,s) equilibrium does not exist.

Let $(s^*(c),S^*(c),T_1^*(c),T_2^*(c))$ denote the cutoff prices and travelling times in the (Q,S,s) equilibrium associated with the menu cost $c\in(0,\overline{c})$. Above we proved that $S^*(c)$ is strictly increasing in c. Since $s^*(c)\equiv(\alpha Q-rc/b)/(2-\alpha)$, $s^*(c)$ is strictly decreasing and $Q-s^*(c)$ is strictly increasing in c. Since $T_1^*(c)\equiv\log(Q/S^*(c))/\pi$ and $T_2^*(c)\equiv\log(S^*(c)/s^*(c))/\pi$, $T_1^*(c)$ is strictly decreasing and $T_2^*(c)$ is strictly increasing in c. Moreover, since $T_1^*(c)+T_2^*(c)$ is strictly increasing, $T_1^*(c)/(T_1^*(c)+T_2^*(c))$ is strictly decreasing in c.

(ii) Using the fact that $S=s\exp(\pi T_2)$ and $s=(\alpha Q-rc/b)/(2-\alpha)$, we can write the equation $\varphi(S,\pi)=0$ as

$$\hat{\varphi}(T_2,\pi)=\left[\frac{1-e^{-(r+\pi)T_2}(1+(r+\pi)T_2)}{(r+\pi)^2T_2}\right]\left[(\alpha Q-rc/b)e^{\pi T_2}-\alpha Q\right] \\ +\left[\frac{1-e^{-(r+\pi)T_2}}{r+\pi}\right]\alpha Q+e^{-rT_2}\left(\frac{\alpha Q}{r}-\frac{c}{b}\right)-\frac{aQ}{r}=0. \tag{B5}$$

After some algebraic transformations, (B5) becomes

$$b\alpha QM(T_2,\pi)=rcN(T_2,\pi), \tag{B6}$$

where the function $M(T_2,\pi)$ is defined as

$$M(T_2,\pi)=-r-\pi(r+\pi)T_2+re^{-(r+\pi)T_2} \\ +(\pi(r+\pi)T_2-r)e^{-rT_2}+re^{\pi T_2}, \tag{B7}$$

and the function $N(T_2,\pi)$ is defined as

$$N(T_2,\pi)=re^{\pi T_2}+(\pi(r+\pi)T_2-r)e^{-T_2}. \tag{B8}$$

Let $M_{T_2}(T_2,\pi)$ and $N_{T_2}(T_2,\pi)$ denote the partial derivatives of $M(T_2,\pi)$ and $N(T_2,\pi)$ with respect to T_2. Similarly, let $M_\pi(T_2,\pi)$ and $N_\pi(T_2,\pi)$ denote the partial derivatives of $M(T_2,\pi)$ and $N(T_2,\pi)$ with respect to π.

Now, let $T_2^*(\pi)$ denote the solution to (B5) with respect to T_2. The derivative of $T_2^*(\pi)$ with respect to the inflation rate π is given by

$$\begin{aligned} T_2^{*\prime}(\pi) &= \frac{M(T_2^*,\pi)N_\pi(T_2^*,\pi)-M_\pi(T_2^*,\pi)N(T_2^*,\pi)}{N(T_2^*,\pi)M_{T_2}(T_2^*,\pi)-N_{T_2}(T_2^*,\pi)M(T_2^*,\pi)} \\ &= \frac{-T_2^*\left\{2-(r+\pi)T_2^*-4e^{-(r+\pi)T_2^*}+\left[2+(r+\pi)T_2^*\right]e^{-2(r+\pi)T_2^*}\right\}}{\pi\left\{1-(r+\pi)T_2^*+\left[(r+\pi)^2T_2^{*2}-2\right]e^{-(r+\pi)T_2^*}+\left[1+(r+\pi)T_2^*\right]e^{-2(r+\pi)T_2^*}\right\}}. \end{aligned} \tag{B9}$$

It is easy to verify that the above expression is strictly negative.

Next, let $S^*(\pi)$ denote the solution to the equation $\varphi(S)=0$ with respect to S, which is given by

$$S^*(\pi)=\frac{\alpha Q-rc/b}{2-\alpha}e^{\pi T_2^*(\pi)}. \tag{B10}$$

The derivative of $S^*(\pi)$ with respect to π has the same sign as the derivative of $\pi T_2^*(\pi)$ with respect to π, which is given by

$$\begin{aligned} &T_2^*(\pi)+\pi T_2^{*\prime}(\pi) \\ &= \frac{T_2^*\left\{e^{-(r+\pi)T_2^*}\left[2+(r+\pi)^2T_2^{*2}\right]-e^{-2(r+\pi)T_2^*}-1\right\}}{1-(r+\pi)T_2^*+\left[(r+\pi)^2T_2^{*2}-2\right]e^{-(r+\pi)T_2^*}+\left[1+(r+\pi)T_2^*\right]e^{-2(r+\pi)T_2^*}}. \end{aligned} \tag{B11}$$

It is easy to verify that the above expression is strictly positive and, hence, $S^*(\pi)$ is strictly increasing in π. Moreover, $S^*(\pi)$ has the following properties: (1) $S^*(\pi)=\alpha Q/(2-\alpha)$ for $\pi\to 0$; (2) $S^*(\pi)>\alpha Q/(2-\alpha)$ for all $\pi>0$; (3) $S^*(\pi)>Q$ for $\gamma\to\infty$. Since $S^*(\pi)$ is a continuous and strictly increasing function of π, the above properties imply that there exists a $\overline{\pi}>0$ such that, for all $\pi\in(0,\overline{\pi})$, $S^*(\pi)\in(\alpha Q/(2-\alpha),Q)$ and, hence, a (Q,S,s) equilibrium exists. In contrast, for $\pi\geq\overline{\pi}$, $S^*(\pi)\geq Q$ and a (Q,S,s) equilibrium does not exist.

Let $(s^*(\pi),S^*(\pi),T_1^*(\pi),T_2^*(\pi))$ denote the cutoff prices and travelling times in the (Q,S,s) equilibrium associated with the inflation rate $\pi\in(0,\overline{\pi})$. We have already established that $S^*(\pi)$ is strictly increasing and $T_2^*(\pi)$ is strictly decreasing in π. Since $s^*(\pi)\equiv(\alpha Q-rc/b)/(2-\alpha)$, $s^*(\pi)$ is independent of π and so is $Q-s^*(\pi)$. Since $T_1^*(\pi)\equiv\log(Q/S^*(c))/\pi$, it follows that $T_1^*(\pi)$ is strictly decreasing in π. Moreover, the ratio $T_1^*(\pi)/(T_1^*(\pi)+T_2^*(\pi))$ is strictly decreasing in π. ‖

C. PROOF OF THEOREM 3

Suppose that $V^*\in(c,b\alpha Q/r]$ is a solution to the equation

$$\begin{aligned} &\left[\frac{1-e^{-(r+\pi)T(V^*)}(1+(r+\pi)T(V^*))}{(r+\pi)^2}\right]\frac{2b(1-\alpha)Q}{T(V^*)} \\ &+\left[\frac{1-e^{-(r+\pi)T(V^*)}}{r+\pi}\right]b\alpha Q+e^{-rT(V^*)}(V^*-c)-V^*=0, \end{aligned} \tag{C1}$$

where

$$T(V^*)\equiv\frac{1}{\pi}\log\left(\frac{b(2-\alpha)Q}{r(V^*-c)}\right).$$

Let S^* be defined as Q. Let s^* be defined as in equation (23). Let F^* be defined as in equation (22) for $s=s^*$ and $S=S^*$. Moreover, let T^* be defined as $\log(S^*/s^*)/\pi$ and let $R(p)$ be defined as in (2) for $F=F^*$. In order to establish that the tuple (F^*,s^*,S^*,V^*) constitutes an (S,s) equilibrium, we need to verify that it jointly satisfies the optimality conditions (3), (4), (6), and (8) and the stationarity conditions (11) and (13). In addition, we need to verify that $s^*\in(0,Q)$ and that F^* is a CDF with support $[s^*,S^*]$.

The tuple (F^*,s^*,S^*,V^*) satisfies the stationarity conditions (11) and (13) by construction. Moreover, notice that F^* is a CDF with support $[s^*,S^*]$ because $F^*(s^*)=0$, $F^*(S^*)=1$ and $F^{*\prime}(p)>0$ for all $p\in[s^*,S^*]$.

The optimality condition (3) is satisfied because $R(s^*)=b(2-\alpha)s^*$ and $s^*=r(V^*-c)/b(2-\alpha)$ imply $R(s^*)=r(V^*-c)$. Moreover, notice that $s^*\in(0,Q)$ because $V^*>c$ implies $s^*>0$ and $V^*\leq b\alpha Q/r$ implies $s^*<Q$. The optimality condition (8) is satisfied because equation (C1) implies

$$\begin{aligned} &\int_0^{T^*}e^{-rt}R(Qe^{-\pi t})dx+e^{-rT^*}(V^*-c)-V^*=0, \\ \iff\quad &V(0)-V^*=0. \end{aligned} \tag{C2}$$

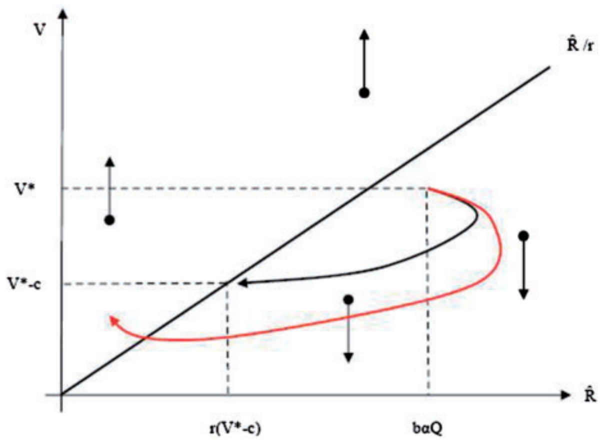

FIGURE 5
Joint Dynamics of $V(t)$ and $\hat{R}(t)$

Now, we need to verify that the tuple (F^*, s^*, S^*, V^*) satisfies the optimality condition (6). That is, we need to verify that the seller's value $V(t)$ is non-greater than V^* for all $t \in [0, T^*]$. To this aim, notice that $V(t)$ satisfies the differential equation

$$\begin{aligned} rV(t) &= \hat{R}(t) + V'(t), \\ \hat{R}(t) &\equiv R(Qe^{-\pi t}). \end{aligned} \tag{C3}$$

The function $\hat{R}(t)$ is given by

$$\hat{R}(t) = e^{-\pi t} bQ \left\{ (2-\alpha) - 2(1-\alpha) \left[1 - \frac{\pi t}{\log(Q/s^*)} \right] \right\}. \tag{C4}$$

The derivative of $\hat{R}(t)$ with respect to t is given by

$$\hat{R}'(t) = \pi e^{-\pi t} bQ \left\{ 2(1-\alpha) \left[1 + \frac{1-\pi t}{\log(Q/s^*)} \right] - (2-\alpha) \right\}. \tag{C5}$$

The derivative $\hat{R}'(t)$ has the same sign as the term in curly brackets in (C5). It is straightforward to verify that this term is strictly increasing in t. Hence, $\hat{R}(t)$ is either strictly decreasing in t over the entire interval $[0, T^*]$, or it is first strictly increasing and then strictly decreasing in t. Notice that $\hat{R}(t)$ cannot be increasing for all $t \in [0, T^*]$ because $\hat{R}(0) = b\alpha Q \geq rV^*$ and $\hat{R}(T^*) = r(V^* - c) < rV^*$.

Consider the phase diagram in Figure 5, which describes the differential equation (C3). The line passing through the origin denotes the locus of points $(\hat{R}, V)$ such that $V = \hat{R}/r$ and, hence, $V' = 0$. Any point below the line is such that $V < \hat{R}/r$ and, hence, $V' < 0$. Any point above the line is such that $V > \hat{R}/r$ and, hence, $V' > 0$. From (8), it follows that $\hat{R}(0) = b\alpha Q$ and $V(0) = V^* \leq b\alpha Q/r$. Hence, the point $(\hat{R}(0), V(0))$ lies either on or below the line and $V'(0) \leq 0$. From (3), it follows that $\hat{R}(T^*) = r(V^* - c)$ and $V(T^*) = V^* - c$. Therefore, the point $(\hat{R}(T^*), V(T^*))$ lies on the line and $V'(T^*) = 0$.

Now, we want to find out the trajectory that the pair $(\hat{R}(t), V(t))$ follows as it travels from the initial point $(b\alpha Q, V^*)$ to the endpoint $(r(V^* - c), V^* - c)$. First, consider the case $V^* < b\alpha Q/r$. In this case, the initial point $(b\alpha Q, V^*)$ lies in the region where $V'(t) < 0$. For as long as $\hat{R}(t)$ increases, $(\hat{R}(t), V(t))$ moves to the south-east of the initial point $(b\alpha Q, V^*)$. When $\hat{R}(t)$ begins to decrease, $(\hat{R}(t), V(t))$ changes direction and moves towards the south-west and, eventually, it crosses the line. Suppose that $(\hat{R}(t), V(t))$ crosses the black line at a time $\hat{T} < T^*$. Then, after time $\hat{T}$, $(\hat{R}(t), V(t))$ moves to the north-west and, since $\hat{R}(t)$ is decreasing and $V(t)$ is increasing, $(\hat{R}(t), V(t))$ does not reach the line again. This contradicts the fact that $(\hat{R}(t), V(t))$ reaches the line at the point $(r(V^* - c), V^* - c)$ at time T^*. Therefore, $(\hat{R}(t), V(t))$ must first cross the line at time T^*. This implies that $V(t)$ is strictly decreasing. Hence, the optimality condition (6) is satisfied. Moreover, $\hat{R}(t) \geq r(V^* - c)$. Hence, the optimality condition (4) is also satisfied.

Second, consider the case $V^* = b\alpha Q/r$. In this case, it is easy to verify that $(\hat{R}(t), V(t))$ must move first to the south-east, then to the south-west and reach the line at time T^*. Also in this case, $V'(t) \leq 0$ for all $t \in [0, T^*]$. This completes the proof that the tuple (F^*, s^*, S^*, V^*) is an (S, s) equilibrium. The proof that there is no (S, s) equilibrium if equation (C1) does not admit a solution for $V^* \in (c, b\alpha Q/r]$ is straightforward and it is omitted for the sake of brevity. ‖

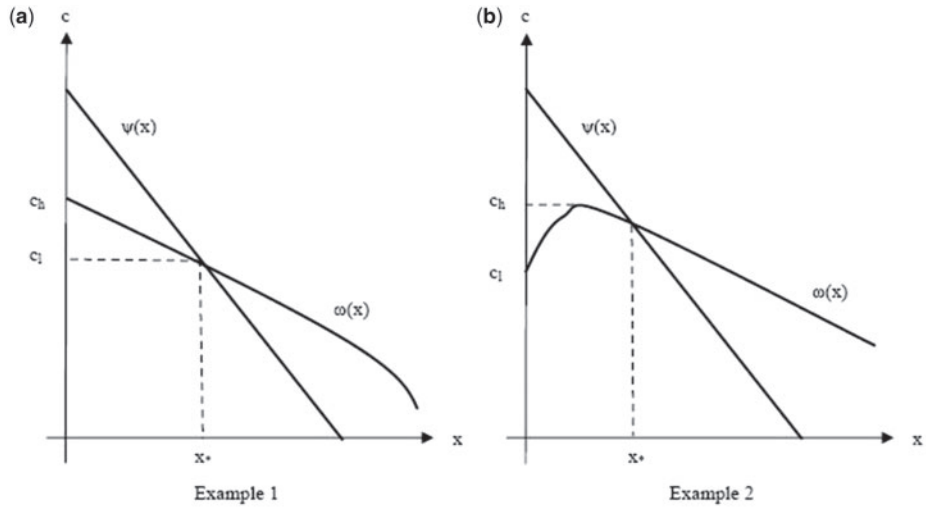

FIGURE 6
Existence of an (S,s) equilibrium

D. PROOF OF THEOREM 4

Theorem 3 states that an (S,s) equilibrium exists if and only if there exists a V^* such that: (a) V^* is a solution to (24); (b) V^* is greater than c and smaller than $b\alpha Q/r$. It is convenient to rewrite these two conditions in terms of $x \equiv V^* - c$ rather than V^*. Condition (a) is then equivalent to x being a solution to

$$\left[\frac{1-e^{-(r+\pi)T(x)}(1+(r+\pi)T(x))}{(r+\pi)^2}\right]\frac{2b(1-\alpha)Q}{T(x)} + \left[\frac{1-e^{-(r+\pi)T(x)}}{r+\pi}\right]b\alpha Q + e^{-rT(x)}x - x = c, \tag{D1}$$

where

$$T(x) \equiv \frac{1}{\pi}\log\left(\frac{b(2-\alpha)Q}{rx}\right).$$

Condition (b) is equivalent to $x>0$ and

$$\frac{b\alpha Q}{r} - x \geq c. \tag{D2}$$

Let $\psi(x)$ denote the left-hand side of (D2). Clearly, the function $\psi(x)$ is such that $\psi(0)=b\alpha Q/r$, $\psi(b\alpha Q/r)=0$ and $\psi'(x)=-1$. Let $\omega(x)$ denote the left-hand side of (D1). It is easy to verify that the function $\omega(x)$ is such that $\omega(0)=b\alpha Q/(r+\pi)$, $\omega'(x)>-1$ and $\omega(x)>0$ for all $x\in[0,b\alpha Q/r]$. Figure 6 illustrates the properties of $\psi(x)$ and $\omega(x)$. These properties guarantee that there exists a unique x^* in the interval $(0,b\alpha Q/r)$ such that $\psi(x^*)=\omega(x^*)$, $\psi(x)>\omega(x)$ for all $x\in[0,x^*)$, and $\psi(x)<\omega(x)$ for all $x\in(x^*,b\alpha Q/r]$. Let c_h denote the maximum of $\omega(x)$ over the interval $[0,x^*]$. Notice that, since $\omega(x)<\psi(x)$ for all $x\in[0,x^*)$ and $\psi(x)\leq b\alpha Q/r$, c_h is strictly smaller than $b\alpha Q/r$. Similarly, let c_ℓ denote the minimum of $\omega(x)$ over the interval $[0,x^*]$. Notice that, since $\omega(x)>0$ for all $x\in[0,x^*]$, c_ℓ is strictly greater than zero. Figure 6 illustrates the definition of x^*, c_ℓ and c_h.

Given the menu cost c, an (S,s) equilibrium is an x such that $\omega(x)=c$ and $\psi(x)\leq c$. If $c<c_\ell$, an (S,s) equilibrium cannot exist because, as one can see from Figure 6, there is no x such that $\omega(x)=c$ and $\psi(x)\leq c$. Similarly, if $c>c_h$, an (S,s) equilibrium cannot exist because, as one can see from Figure 6, there is no x such that $\omega(x)=c$ and $\psi(x)\leq c$. In contrast, if $c\in[c_\ell,c_h]$, there exists at least one (S,s) equilibrium. There may exist multiple (S,s) equilibria because the function $\omega(x)$ may not be monotonic. This case is illustrated in Figure 6b. Intuitively, multiplicity may arise because of a feed-back effect between the cross-sectional price distribution F and the maximum present value of seller's profits V^*. When V^* is higher, the point s at which a seller finds it optimal to reset its nominal price is higher. In turn, this implies that the cross-sectional price distribution F is higher (in the sense of first-order stochastic dominance). Conversely, when the cross-sectional price distribution F is higher, the competition faced by a seller with a given price is lower. In turn, this implies that the maximum present value of the seller's profits V^* is higher. For some parameter values, this feed-back effect between V^* and F can be so strong as to generate multiple equilibria.

Finally, note that the interval $[c_\ell, c_h]$ of menu costs for which an (S,s) equilibrium exists always includes the upper bound of the interval $[0,\overline{c}]$ of menu costs for which a (Q,S,s) equilibrium exists. In fact, if and only if $c=\omega(x^*)$, there exists an (S,s) equilibrium in which the maximum present value of the seller's profits is $b\alpha Q/r$. And if and only if $c\to\overline{c}$, the (Q,S,s) equilibrium converges to an (S,s) equilibrium in which the maximum present value of the seller's profits is $b\alpha Q/r$. Therefore, $\overline{c}=\omega(x^*)$ and $\omega(x^*)\in[c_\ell,c_h]$. ‖

E. DIVISIBLE GOOD

E.1. *(Q,S,s) Equilibrium*

We start by solving for the equilibrium objects when sellers follow a (Q,S,s) pricing rule. For $t=0$, the equilibrium condition (7) implies $R(Q)=rV^*$. Since $R(Q)=b\alpha Q^\theta$, we can solve $R(Q)=rV^*$ with respect to V^* and find

$$V^*=b\alpha Q^\theta \big/ r. \tag{E1}$$

For $t\in[0,T_1]$, the equilibrium condition (7) implies $R(p)=rV^*$ for all $p\in[S,Q]$ Since $R(p)$ is equal to $b[\alpha+2(1-\alpha)(1-F(p))]p^\theta$ and V^* is equal to (E1), we can solve $R(p)=rV^*$ with respect to the price distribution F. We find that

$$F(p)=1-\frac{\alpha}{2(1-\alpha)}\frac{Q^\theta-p^\theta}{p^\theta},\ \forall p\in[S,Q]. \tag{E2}$$

The stationarity condition (11) is an equation for the derivative of the price distribution F over the interval (s,S). Integrating (11) and using the boundary conditions $F(s)=0$ and $F(S-)=F(S+)$, we find that

$$F(p)=\left[1-\frac{\alpha}{2(1-\alpha)}\frac{Q^\theta-S^\theta}{S^\theta}\right]\frac{\log p-\log s}{\log S-\log s},\forall p\in[s,S]. \tag{E3}$$

The stationarity condition (12) is an equation for the derivative of the price distribution F over the interval (S,Q). Using the fact that the price distribution F is given by (15), we can solve for (12) with respect to the distribution of new prices G and find

$$G(p)=1-\left(1-\frac{\alpha}{2(1-\alpha)}\frac{Q^\theta-S^\theta}{S^\theta}\right)^{-1}\frac{\alpha\log(S/s)Q^\theta}{2(1-\alpha)p^\theta},\ \forall p\in(S,Q). \tag{E4}$$

The distribution G must have a mass point of measure $\chi(S)=G(S)$ at S and a mass point of measure $\chi(Q)=1-G(Q-)$, where $\chi(S)$ and $\chi(Q)$ are given by

$$\chi(S)=1-\left(1-\frac{\alpha}{2(1-\alpha)}\frac{Q^\theta-S^\theta}{S^\theta}\right)^{-1}\frac{\alpha\log(S/s)Q^\theta}{2(1-\alpha)S^\theta}, \tag{E5}$$

$$\chi(Q)=\left(1-\frac{\alpha}{2(1-\alpha)}\frac{Q^\theta-S^\theta}{S^\theta}\right)^{-1}\frac{\alpha\log(S/s)}{2(1-\alpha)}. \tag{E6}$$

The optimality condition (3) implies $R(s)=r(V^*-c)$. Since $R(s)=b(2-\alpha)s^\theta$ and V^* is given by (E1), we can solve $R(s)=r(V^*-c)$ with respect to s and find

$$s=\left(\frac{\alpha Q^\theta-rc/b}{2-\alpha}\right)^{1/\theta}. \tag{E7}$$

The optimality condition (8) states that a seller with a real price of S must attain the maximized present value of profits V^*. After substituting out F, G and V^* and solving the integral, we can rewrite (8) as one equation in the one unknown V^*. That is,

$$\begin{aligned}\varphi(S)\equiv\quad &\left[\frac{1-e^{-(r+\pi\theta)T_2(S)}(1+(r+\pi\theta)T_2(S))}{(r+\pi\theta)^2}\right]\frac{(2-\alpha)S^\theta-\alpha Q^\theta}{T_2(S)}\\ &+\left[\frac{1-e^{-(r+\pi\theta)T_2(S)}}{r+\pi\theta}\right]\alpha Q^\theta-\left[\frac{1-e^{-rT_2(S)}}{r}\right]\alpha Q^\theta-e^{-rT_2(S)}c/b=0,\end{aligned} \tag{E8}$$

where

$$T_2(S)\equiv\frac{\log(S/s)}{\pi},\quad s=\left(\frac{\alpha Q^\theta-rc/b}{2-\alpha}\right)^{1/\theta}.$$

A (Q,S,s) equilibrium exists only if the equation $\varphi(S)$ admits a solution for some S in the interval (s,Q). It is easy to verify that $\varphi(S)$ is strictly increasing and takes on strictly negative values strictly negative for all $S\in[s,(\alpha Q^\theta/(2-\alpha))^{1/\theta}]$.

Therefore, a (Q,S,s) equilibrium exists only if the equation $\varphi(Q)>0$. Following the same proof as in Appendix A, we can show that $\varphi(Q)>0$ is also a sufficient condition for the existence of a (Q,S,s) equilibrium and we can characterize the properties of $V(t)$ and $R(p)$. Following the same proof as in Appendix B, we can show that $\varphi(Q)>0$ if and only if the menu cost is smaller than some $\overline{c}>0$.

Our findings are summarized in the following theorem.

Theorem E1. *(1) A (Q,S,s) equilibrium exists iff $\varphi(Q)>0$. If a (Q,S,s) equilibrium exists, it is unique and: (a) $V'(t)=0$ for all $t\in(0,T_1)$ and $V'(t)<0$ for all $t\in(T_1,T_1+T_2)$; (2) $\hat{R}'(t)=0$ for all $t\in(0,T_1)$, $\hat{R}'(t)>0$ for all $t\in(T_1,\hat{T})$ and $\hat{R}'(t)<0$ for all $t\in(T,T_1+T_2)$, where $\hat{R}(t)\equiv R(p(t))$ and $\hat{T}\in(T_1,T_1+T_2)$. (ii) The condition $\varphi(Q)>0$ is satisfied if and only if $c\in(0,\overline{c})$, with $\overline{c}>0$.*

E.2. *(S,s) Equilibrium*

We now turn to the (S,s) equilibrium. The stationarity condition (11) is an equation for the derivative of the price distribution F over the interval (s,S). Using the fact that $S=Q$ and the boundary conditions $F(s)=0$ and $F(Q)=1$, we can integrate (11) and find that

$$F(p)=\frac{\log p-\log s}{\log Q-\log s},\ \forall p\in(s,Q). \tag{E9}$$

The optimality condition (3) implies $R(s)=r(V^*-c)$. Since $R(s)=b(2-\alpha)s^{\theta}$, we can solve the optimality condition with respect to s and find

$$s=\left(\frac{r(V^*-c)}{b(2-\alpha)}\right)^{1/\theta}. \tag{E10}$$

The optimality condition (8) implies that a seller with a real price of S must attain the maximized present value of profits V^*. Using the fact that S is equal to Q, that F is given by (22) and that s is given by (23), we can rewrite (8) as one equation in the unknown V^*. Specifically, we can rewrite it as

$$\begin{aligned}\vartheta(V^*)\equiv&\left[\frac{1-e^{-(r+\pi\theta)T}(1+(r+\pi\theta)T)}{(r+\pi\theta)^2}\right]\frac{2b(1-\alpha)Q^{\theta}}{T}\\&+\left[\frac{1-e^{-(r+\pi\theta)T}}{r+\pi\theta}\right]b\alpha Q^{\theta}+e^{-rT}(V^*-c)-V^*=0,\end{aligned} \tag{E11}$$

where

$$T\equiv\frac{1}{\pi}\log(Q/s)\text{ and }s=\left(\frac{r(V^*-c)}{b(2-\alpha)}\right)^{1/\theta}.$$

An (S,s) equilibrium exists only if the equation $\vartheta(S)$ admits a solution for some V^* in the interval $(c,\alpha bQ^{\theta}/r]$. It is immediate to verify that the condition is necessary. Following the same proof as in Appendix C, we can show that the condition is also sufficient. Following the same proof as in Appendix D, we can show that the condition is satisfied if and only if the menu cost is greater than c_ℓ and smaller than c_h, with $c_\ell>0$ and $c_h<\alpha bQ^{\theta}/r$.

Our findings are summarized in the following theorem.

Theorem E2. *(1) An (S,s) equilibrium exists iff $\vartheta(V^*)=0$ for some $V^*\in(c,b\alpha Q^{\theta}/r]$. (2) An (S,s) equilibrium exists iff $c\in[c_\ell,c_h]$, with $0<c_\ell<c_h<b\alpha Q/r$ and $c_\ell\leq\overline{c}\leq c_h$.*

Acknowledgments. We are grateful to the editor, Dimitri Vayanos, and three anonymous referees. We also thank Fernando Alvarez, Manuel Amador, Roland Bénabou, Ben Eden, Allen Head, Greg Kaplan, Francesco Lippi, Moritz Meyer-ter-Vehn, Pierre-Olivier Weill and, especially, Randy Wright for useful discussions on early versions of the article. We benefitted from the comments of our audiences at Princeton, Stanford, Chicago, NYU, UCLA, Wisconsin-Madison, Carnegie Mellon, Michigan, UBC, Simon Fraser, UCL, EIEF, Universita' Bocconi, Ohio State, the Econometric Society Meeting (Evanston 2012), the Search and Matching Workshop (Philadelphia 2012), the Frontiers of Macroeconomics Conference at Queen's University (Kingston 2016).

REFERENCES

ALVAREZ, F. and LIPPI, F. (2014), "Price Setting with Menu Cost for Multiproduct Firms", *Econometrica*, **82**, 89–135.

ALVAREZ, F., LE BIHAN, H. and LIPPI, F. (2016), "The Real Effect of Monetary Shocks in Sticky Price Models: A Sufficient Statistic Approach", *American Economic Review*, Forthcoming.

BASU, S. and FERNALD, J. (1997), "Returns to Scale in U.S. Production: Estimates and Implications", *Journal of Political Economy*, **105**, 249–283.
BÉNABOU, R. (1988), "Search, Price Setting and Inflation", *Review of Economic Studies*, **55**, 353–376.
BÉNABOU, R. (1989), "Optimal Price Dynamics and Speculation with a Storable Good", *Econometrica*, **57**, 41–80.
BÉNABOU, R. (1992), "Inflation and Efficiency in Search Markets", *Review of Economic Studies*, **59**, 299–329.
BURDETT, K. and JUDD, K. (1983), "Equilibrium Price Dispersion", *Econometrica*, **51**, 955–970.
BUTTERS, G. (1977), "Equilibrium Distributions of Sales and Advertising Prices", *Review of Economic Studies*, **44**, 465–491.
CAPLIN, A. and LEAHY, J. (1997), "Aggregation and Optimization with State-Dependent Pricing", *Econometrica*, **65**, 601–625.
CAPLIN, A. and LEAHY, J. (2010), "Economic Theory and the World of Practice: A Celebration of the (S,s) Model", *Journal of Economic Perspectives*, **24**, 183–202.
CAPLIN, A. and SPULBER, D. (1987), "Menu Costs and the Neutrality of Money", *Quarterly Journal of Economics*, **102**, 703–725.
DE LOECKER, J. and WARZYNSKI, F. (2012), "Markups and Firm-Level Export Status", *American Economic Review*, **102**, 2437–2471.
DIAMOND, P. (1971), "A Model of Price Adjustment", *Journal of Economic Theory*, **3**, 156–168.
DIAMOND, P. (1993), "Search, Sticky Prices, and Inflation", *Review of Economic Studies*, **60**, 53–68.
DOTSEY, M., KING, R. and WOLMAN, A. (1999), "State-Dependent Pricing and the General Equilibrium Dynamics of Money and Output", *Quarterly Journal of Economics*, **114**, 655–690.
EDEN, B. (1994), "The Adjustment of Prices to Monetary Shocks When Trade is Uncertain and Sequential", *Journal of Political Economy*, **102**, 493–509.
GOLOSOV, M. and LUCAS, R. (2007), "Menu Costs and Phillips Curves", *Journal of Political Economy*, **115**, 171–99.
HEAD, A., LIU, L., MENZIO, G., *et al*. (2012), "Sticky Prices: A New Monetarist Approach", *Journal of the European Economic Association*, **10**, 939–973.
KAPLAN, G. and MENZIO, G. (2015), "The Morphology of Price Dispersion", *International Economic Review*, **56**, 1165–1206.
KAPLAN, G. and MENZIO, G. (2016), "Shopping Externalities and Self-Fulfilling Unemployment Fluctuations", *Journal of Political Economy*, **124**, 771–825.
KAPLAN, G., MENZIO, G., RUDANKO, L., *et al*. (2016), "Relative Price Dispersion: Evidence and Theory" (NBER Working Paper 21931).
KLENOW, P. and KRYVTSOV, O. (2008), "State-Dependent or Time-Dependent Pricing: Does it Matter for Recent U.S. Inflation?" *Quarterly Journal of Economics*, **123**, 863–904.
KLETTE, T. (1999), "Market Power, Scale Economies and Productivity: Estimates from a Panel of Establishment Data", *Journal of Industrial Organization*, **27**, 451–476.
LACH, S. and TSIDDON, D. (1992), "The Behavior of Prices and Inflation: An Empirical Analysis of Disaggregated Data", *Journal of Political Economy*, **100**, 349–389.
LAGOS, R. and WRIGHT, R. (2005), "A Unified Framework for Monetary Theory and Policy Analysis", *Journal of Political Economy*, **113**, 463–484.
LUCAS, R. and STOKEY, N. (1987), "Money and Interest in a Cash-in-Advance Economy", *Econometrica*, **55**, 491–513.
MENZIO, G. and TRACHTER, N. (2016), "Equilibrium Price Dispersion with Sequential Search", *Journal of Economic Theory*, **160**, 188–215.
MIDRIGAN, V. (2011), "Menu Costs, Multi-Product Firms, and Aggregate Fluctuations", *Econometrica*, **79**, 1139–1180.
NAKAMURA, E. and STEINSSON, J. (2008), "Five Facts About Prices: A Reevaluation of Menu Cost Models", *Quarterly Journal of Economics*, **123**, 1415–1464.
PRESCOTT, E. (1975), "Efficiency of the Natural Rate", *Journal of Political Economy*, **83**, 1229–1236.
SHESHINSKI, E. and WEISS, Y. (1977), "Inflation and Costs of Price Adjustment", *Review of Economic Studies*, **44**, 287–303.
SORENSEN, A. (2000), "Equilibrium Price Dispersion in Retail Markets for Prescription Drugs", *Journal of Political Economy*, **108**, 833–850.
VARIAN, H. (1980), "A Theory of Sales", *American Economic Review*, **70**, 651–659.