Rolodex Game in Networks*

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Abstract

The paper studies the Rolodex bargaining game in networks. We first revisit the Rolodex game originally proposed in the context of intra firm bargaining, in which a central player bargains sequentially with multiple peripheral player. We show that the unique no-delay SPE of this game yields the Myerson-Shapley value for the star graph in which the central player is linked to all peripheral players. Second, we propose a Rolodex game for a general graph. Links in this graph negotiate sequentially, with one of the linked players making an offer to the other. If the respondent rejects, the link moves to the end of the line and the direction of the offer is reversed for the next negotiation of this link. As in the original Rolodex game, all agreements are renegotiated in the event of a breakdown. We show that the unique no-delay SPE of this game yields the Myerson-Shapley value for the corresponding graph.

JEL Codes:

Keywords: Non-cooperative bargaining; Rolodex game; Myerson-Shapley value.

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1 Value function for a Network and Myerson-Shapley value

We follow de Fontenay and Gans (2014) in adopting the specification of the value function for a network introduced in Navarro (2006). Let N denote the set of players and let $\mathcal{N}^u$ denote an undirected graph. Let $C(\mathcal{N}^u)$ the set of components of $\mathcal{N}^u$. Players in a component are linked directly or indirectly to each other but not to any other players, thus $C(\mathcal{N}^u)$ is a partition of N. The Myerson-Shapley value for the network $\mathcal{N}^u$ is characterized by the properties of *fair allocation* and *component balance*. Let $\Psi_i(\mathcal{N}^u)$ denote the Myerson-Shapley value of player i. Fair allocation is a recursive property which requires that

$$\Psi_i(\mathcal{N}^u) - \Psi_i(\mathcal{N}^u - ij) = \Psi_j(\mathcal{N}^u) - \Psi_j(\mathcal{N}^u - ij). \tag{1}$$

for every link $ij \in \mathcal{N}^u$ and every graph $\mathcal{N}^u$, that is, both players involved in a link lose the same amount if this link is severed.

Component balance requires that

$$\sum_{i \in h} \Psi_i(\mathcal{N}^u) = v(h, \mathcal{N}^u) \tag{2}$$

for all components $h \in C(\mathcal{N}^u)$ and graphs $\mathcal{N}^u$. Here v is the value function that assigns a value to each component of a graph, for all graphs. Notice that the value assigned to a component is allowed to depend on the entire graph.

2 Star Graph

2.1 Extensive Form

In this section we consider the Rolodex game proposed by Brügemann, Menzio, and Gautier (2017, BMG) in the setting of intra firm bargaining. In this setting a firm bargains sequentially with multiple workers. The value function associated with this game has a special structure as the firm is assumed to be essential in production. BGM find that in this setting the Rolodex game yields the Shapley value for each player. Here we allow for a general value function while maintaining the same extensive form. In particular, there is central player that bargains sequentially with multiple peripheral players.

We start by introducing some notation. We use c as the index of the central player. We describe the extensive form in a recursive way. Let $o \in \{c, p\}$ denote the type of player that makes the next offer, where c stands for central and p for peripheral. At a given point in the game there is a list of remaining links $\mathcal{N}$. For consistency with the notation used for the game with a general graph in Section 3, we write this as a list of directed links from the peripheral player to the central player, in the order in which the central player will negotiate with these players when negotiations start from scratch with this graph. For example, if players 2 and 3 are the remaining peripheral players, we have $\mathcal{N} = \{21, 31\}$. There is a second list $\mathcal{K}$ of directed links, which is the subset of links from $\mathcal{N}$ which have not yet reached an agreement, in the order in which the central player will negotiate with the peripheral players. Finally there is also a set of agreements $\mathcal{A}$. Each agreement is a pair consisting of the index of the peripheral player involved in the agreement and the transfer the central player has agreed to make to this peripheral player. The state of the game is described by $(o, \mathcal{N}, \mathcal{K}, \mathcal{A})$.

The game ends when $\mathcal{K}$ is empty, that is, when all peripheral players that remain connected to the central player have reached an agreement. Let h denote the component of $\mathcal{N}$ containing the central player. The payoff of the central player is

$$v(h, \mathcal{N}^u) - \sum_{(i,T_i)\in\mathcal{A}} T_i$$

where $\mathcal{N}^u$ denotes the graph of undirected links corresponding to the list $\mathcal{N}$. A peripheral player $i \in h$ receives the payoff T_i for which $(i, T_i) \in \mathcal{A}$. A peripheral player $i \notin h$ has become disconnected from c and forms his own component, receiving $v(i, \mathcal{N}^u)$.

If $\mathcal{K}$ is non-empty, then the game continues. The next link to negotiate is $\mathcal{K}_1$. Let $\mathcal{K}_{1p}$ denote the peripheral player involved in this link. If $o = p$, then $\mathcal{K}_{1p}$ makes an offer T to c. If c accepts, then the link $\mathcal{K}_1$ is removed from the list without agreements, the agreement $(\mathcal{K}_{1p}, T)$ is added to $\mathcal{A}$, and the game continues in state $[p, \mathcal{N}, \mathcal{K} - \mathcal{K}_1, \mathcal{A} + (\mathcal{K}_{1p}, T)]$. If c rejects, then with probability $q_{\#\mathcal{K}}$ there is a breakdown. Notice that the breakdown probability is allowed to depend on the number of links without agreement, for reasons to be discussed further below. If there is no breakdown, then the game continues in state $(c, \mathcal{N}, \mathcal{K}, \mathcal{A})$. If there is a breakdown, then the link $\mathcal{K}_1$ is deleted from $\mathcal{N}$ and all agreements in $\mathcal{A}$ are deleted. The new order of negotiation is given by $\mathcal{N}$ after the deletion of $\mathcal{K}_1$. Thus the game continues in state $(\mathcal{N} - \mathcal{K}_1, \mathcal{N} - \mathcal{K}_1, \emptyset)$.

If $o = c$, then c makes an offer T to $\mathcal{K}_{1p}$. If $\mathcal{K}_{1p}$ accepts, then the game continues in state $[p, \mathcal{N}, \mathcal{K} - \mathcal{K}_1, \mathcal{A} + (\mathcal{K}_{1p}, T)]$. If $\mathcal{K}_{1p}$ rejects, then with probability $q_{\#\mathcal{K}}$ there is breakdown. If there is no breakdown, then link $\mathcal{K}_1$ rotates to the end of the list. Let $\mathcal{K}^R$ denote the resulting new list. The game then continues in state $\left(p, \mathcal{N}, \mathcal{K}^R, \mathcal{A}\right)$. If there is a breakdown, the game continues in state $(\mathcal{N} - \mathcal{K}_1, \mathcal{N} - \mathcal{K}_1, \emptyset)$.

We are interested in the limit as breakdown probabilities approach zero. The role of breakdown is to act as a force towards agreement that ensures uniqueness of no-delay SPE. This force is only needed in situations in which a single link remains without agreement, since otherwise the prospect for a link to rotate to the end of the list already acts as such a force. Thus without loss of generality we set $q_k = 0$ for $k > 1$. Furthermore, notice that for situations with $\#\mathcal{K} = 1$ the extensive form coincides with that of the Binmore-Rubinstein-Wolinsky bilateral alternating offers game with risk of breakdown, which is known to have a unique SPE if the breakdown probability is strictly positive. For simplicity, here we directly focus on the limiting payoffs as the breakdown probability approaches zero.

2.2 No-delay SPE

Let $N(\mathcal{K}^u)$ denote the set of players involved in a link in $\mathcal{K}^u$, including c unless $\mathcal{K}^u$ is empty. We start by specifying a function that is useful in characterizing equilibrium payoffs. The arguments of this function are the undirected graphs $\mathcal{N}^u$ and $\mathcal{K}^u$ and the set of agreements $\mathcal{A}$. For all $i \in N(\mathcal{K}^u)$, recursively define $\tilde{\Phi}_i(\mathcal{N}^u, \mathcal{K}^u, s)$ via the surplus splitting property

$$\tilde{\Phi}_c(\mathcal{N}^u, \mathcal{K}^u, \mathcal{A}) - \tilde{\Phi}_c(\mathcal{N}^u - ic, \mathcal{N}^u - ic, \emptyset) = \tilde{\Phi}_i(\mathcal{N}^u, \mathcal{K}^u, \mathcal{A}) - v(i, \mathcal{N}^u - ic) \tag{3}$$

for all $i \in N(\mathcal{K}^u) - c$, and the adding-up property

$$\sum_{i \in N(\mathcal{K}^u)} \tilde{\Phi}_i(\mathcal{N}^u, \mathcal{K}^u, \mathcal{A}) = v(N(\mathcal{K}^u), \mathcal{N}^u) - \sum_{(j, T_j) \in \mathcal{A}} T_j. \tag{4}$$

It follows immediately that

$$\tilde{\Phi}_i(\mathcal{N}^u, \mathcal{N}^u, \emptyset) = \Psi_i(\mathcal{N}^u)$$

since equations (3) and (4) coincide with the properties of fair allocation in equation (1) and component balance in equation (2) for this case, respectively.

We will show that no-delay SPE payoffs in state $(o, \mathcal{N}, \mathcal{K}, \mathcal{A})$ are unique and given by

$$\Phi_i(o, \mathcal{N}, \mathcal{K}, \mathcal{A}) = \tilde{\Phi}_i(\mathcal{N}^u, \mathcal{K}^u, \mathcal{A}). \tag{5}$$

Thus the order of peripheral players and the type of the offering player does not affect equilibrium payoffs, and in states $(p, \mathcal{N}, \mathcal{N}, \emptyset)$ in which negotiations start from scratch, equilibrium payoffs are the Myerson-Shapley values for the star graph $\mathcal{N}^u$.

We will verify this claim by induction over the number of peripheral players that have yet to reach agreement and the number of peripheral players remaining in the game. The statement is true for the base case in which there is only one peripheral player without agreement, since in that case the payoffs prescribed by equations (3)–(4) coincide with the payoffs in the BRW game. In the induction step, we consider states $(o, \mathcal{N}, \mathcal{K}, \mathcal{A})$ in which at least two links are without agreement, and take as given that the induction hypothesis is true for states in which fewer peripheral players remain connected to c, and for states in which the the number of connected peripheral players is the same but more have already reached an agreement.

Let $m_i(o, \mathcal{N}, \mathcal{K}, \mathcal{A})$ and $M_i(o, \mathcal{N}, \mathcal{K}, \mathcal{A})$ denote the infimum and supremum payoff of player i in state $(o, \mathcal{N}, \mathcal{K}, \mathcal{A})$ across all no-delay SPEs.

Consider state $(c, \mathcal{N}, \mathcal{K}, \mathcal{A})$ starting with an offer T by c to $\mathcal{K}_{1p}$. If $\mathcal{K}_{1p}$ rejects, the game enters state $\left(p, \mathcal{N}, \mathcal{K}^R, \mathcal{A}\right)$. Peripheral player $\mathcal{K}_{1p}^R$ then makes an offer T^R that is accepted by c. After this the game enters state

$$\left[p, \mathcal{N}, \mathcal{K} - \mathcal{K}_1^R, \mathcal{A} + \left(\mathcal{K}_{1p}^R, T^R\right)\right].$$

Since one additional peripheral player has an agreement, payoffs in this state are given by the induction hypothesis. In particular, the payoff of player $\mathcal{K}_{1p}$ is

$$\tilde{\Phi}_{\mathcal{K}_{1p}}\left[\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1^R, \mathcal{A} + \left(\mathcal{K}_{1p}^R, T^R\right)\right]$$

which is decreasing in T^R. The equilibrium payoff of $\mathcal{K}_{1p}^R$ satisfies $T^R \geq m_{\mathcal{K}_{1p}^R}(p, \mathcal{N}, \mathcal{K}^R, \mathcal{A})$. Thus the rejection payoff of $\mathcal{K}_{1p}$ is at most

$$\tilde{\Phi}_{\mathcal{K}_{1p}}\left\{\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1^R, \mathcal{A} + \left[\mathcal{K}_{1p}^R, m_{\mathcal{K}_{1p}^R}(p, \mathcal{N}, \mathcal{K}^R, \mathcal{A})\right]\right\}.$$

In equilibrium $\mathcal{K}_{1p}$ must accept any offer strictly above this upper bound. Thus c will not offer strictly more than this upper bound since strictly lower offers would also be accepted. Furthermore, if an offer T is accepted by $\mathcal{K}_{1p}$, then the game enters state

$$[p, \mathcal{N}, \mathcal{K} - \mathcal{K}_1, \mathcal{A} + (\mathcal{K}_{1p}, T)]$$

Since there is an additional agreement in this state, payoffs are given by the induction hypothesis. In particular the payoff of c is

$$\tilde{\Phi}_c\left[\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1, \mathcal{A} + (\mathcal{K}_{1p}, T)\right]$$

which is decreasing in T. Thus we obtain the following lower bound on the payoff of c

$$\begin{aligned} & m_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A}) \\ & \geq \tilde{\Phi}_c\left[\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1, \mathcal{A} + \left(\mathcal{K}_{1p}, \tilde{\Phi}_{\mathcal{K}_{1p}}\left\{\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1^R, \mathcal{A} + \left[\mathcal{K}_{1p}^R, m_{\mathcal{K}_{1p}^R}(p, \mathcal{N}, \mathcal{K}^R, \mathcal{A})\right]\right\}\right)\right]. \end{aligned} \tag{6}$$

Analogously [ADD MORE DETAILS]

$$\begin{aligned} & M_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A}) \\ & \leq \tilde{\Phi}_c\left[\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1, \mathcal{A} + \left(\mathcal{K}_{1p}, \tilde{\Phi}_{\mathcal{K}_{1p}}\left\{\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1^R, \mathcal{A} + \left[\mathcal{K}_{1p}^R, M_{\mathcal{K}_{1p}^R}(p, \mathcal{N}, \mathcal{K}^R, \mathcal{A})\right]\right\}\right)\right]. \end{aligned} \tag{7}$$

Next, consider state $(p, \mathcal{N}, \mathcal{K}, \mathcal{A})$ in which the next action is an offer T by $\mathcal{K}_{1p}$ to c. If c rejects, then c gets to make a counteroffer to $\mathcal{K}_{1p}$ and gets at least $m_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A})$. If c accepts T, then the game enters state $[p, \mathcal{N}, \mathcal{K} - \mathcal{K}_1, \mathcal{A} + (\mathcal{K}_{1p}, T)]$. Since there is an additional agreement in this state, payoffs are given by the induction hypothesis. In particular, the central player obtains

$$\tilde{\Phi}_c \left[\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1, \mathcal{A} + (\mathcal{K}_{1p}, T)\right].$$

Considering this as a function of T, let

$$\tilde{\Phi}_c^{-1} \left(\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1, \mathcal{K}_{1p}, x\right)$$

denote its inverse. We then obtain the following upper bound for the payoff of $\mathcal{K}_{1p}$:

$$M_{\mathcal{K}_{1p}} (p, \mathcal{N}, \mathcal{K}, \mathcal{A}) \leq \tilde{\Phi}_c^{-1} \left[\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1, \mathcal{K}_{1p}, m_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A})\right]. \tag{8}$$

Similarly [ADD MORE DETAILS]

$$m_{\mathcal{K}_{1p}} (p, \mathcal{N}, \mathcal{K}, \mathcal{A}) \geq \tilde{\Phi}_c^{-1} \left[\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1, \mathcal{K}_{1p}, M_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A})\right]. \tag{9}$$

Starting with inequality (6) and repeatedly substituting, in this order, inequalities (9), (7), (8) and again (6) while cycling through the order of peripheral players until arriving again at $m_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A})$, we obtain an inequality of the form

$$m_c (c, \mathcal{N}, \mathcal{K}, \mathcal{A}) \geq \alpha (c, \mathcal{N}, \mathcal{K}, \mathcal{A}) + \beta (c, \mathcal{N}, \mathcal{K}, \mathcal{A}) \, m_c (c, \mathcal{N}, \mathcal{K}, \mathcal{A})$$

for appropriately defined coefficients $\alpha (c, \mathcal{N}, \mathcal{K}, \mathcal{A})$ and $\beta (c, \mathcal{N}, \mathcal{K}, \mathcal{A})$. It is easy to see that $\beta (c, \mathcal{N}, \mathcal{K}, \mathcal{A}) \in (0, 1)$, so we obtain

$$m_c (c, \mathcal{N}, \mathcal{K}, \mathcal{A}) \geq [1 - \beta(c, \mathcal{N}, \mathcal{K}, \mathcal{A})]^{-1} \alpha(c, \mathcal{N}, \mathcal{K}, \mathcal{A})$$

with the identical coefficients $\alpha (c, \mathcal{N}, \mathcal{K}, \mathcal{A})$ and $\beta (c, \mathcal{N}, \mathcal{K}, \mathcal{A})$. Proceeding analogously but starting with inequality (7),we obtain

$$M_c (c, \mathcal{N}, \mathcal{K}, \mathcal{A}) \leq [1 - \beta(c, \mathcal{N}, \mathcal{K}, \mathcal{A})]^{-1} \alpha(c, \mathcal{N}, \mathcal{K}, \mathcal{A}).$$

Thus $m_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A}) = M_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A})$ and (6)–(7) hold as equalities. Thus no-delay SPE payoffs of offer-making players $\Phi_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A})$ and $\Phi_{\mathcal{K}_{1p}}(p, \mathcal{N}, \mathcal{K}, \mathcal{A})$ across the $\#N (\mathcal{K}^u) - 1$ lists $\mathcal{K}$ that the game is rotating through are uniquely determined by the system of $2 \cdot (\#N (\mathcal{K}^u) - 1)$ equations

$$\begin{aligned} &\Phi_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A}) \\ &= \tilde{\Phi}_c \left[\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1, \mathcal{A} + \left(\mathcal{K}_{1p}, \tilde{\Phi}_{\mathcal{K}_{1p}} \left\{\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1^R, \mathcal{A} + \left[\mathcal{K}_{1p}^R, \Phi_{\mathcal{K}_{1p}^R}(p, \mathcal{N}, \mathcal{K}^R, \mathcal{A})\right]\right\}\right)\right], \end{aligned} \tag{10}$$

$$\begin{aligned} &\Phi_{\mathcal{K}_{1p}} (p, \mathcal{N}, \mathcal{K}, \mathcal{A}) \\ &= \tilde{\Phi}_c^{-1} \left[\mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1, \mathcal{K}_{1p}, \Phi_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A})\right].,\end{aligned} \tag{11}$$

which in turn implies that all equilibrium payoffs are uniquely determined. [HERE WE HAVE SHOWN UNIQUENESS BUT NOT YET EXISTENCE. ADD BRIEF DISCUSSION OF EXISTENCE, WHICH REQUIRES NON-NEGATIVE GAINS FROM TRADE FOR LINKS.]

Having established that no-delay SPE payoffs are unique, we now verify that they satisfy the induction hypothesis. We do this by writing down the system of linear equations satisfied by these payoffs and verify that it is satisfied by the payoffs prescribed by the induction hypothesis.

We will solve for $\Phi_i(o, \mathcal{N}, \mathcal{K}, \mathcal{A})$ for all $\#N(\mathcal{K}^u)$ players, the two offers types $o \in \{c, p\}$, and all $\#N(\mathcal{K}^u) - 1$ possible lists $\mathcal{K}$ that the game is rotating through, for given $\mathcal{N}$ and $\mathcal{A}$. Thus overall we need to solve for $2 \cdot \#N(\mathcal{K}^u) \cdot (\#N(\mathcal{K}^u) - 1)$ payoffs.

According to equation (10), in state $(c, \mathcal{N}, \mathcal{K}, \mathcal{A})$ the central player offers

$$\tilde{\Phi}_{\mathcal{K}_{1p}} \left\{ \mathcal{N}^u, \mathcal{K}^u - \mathcal{K}_1^R, \mathcal{A} + \left[\mathcal{K}_{1p}^R, \Phi_{\mathcal{K}_{1p}^R}(p, \mathcal{N}, \mathcal{K}^R, \mathcal{A}) \right] \right\}$$

and this offer is accepted by $\mathcal{K}_{1p}$. By definition this offer is equal to the payoff $\mathcal{K}_{1p}$ receives in state $\left(p, \mathcal{N}, \mathcal{K}^R, \mathcal{A}\right)$ in which $\mathcal{K}_{1p}$ is at the end of the line. In other words, when $\mathcal{K}_{1p}$ receives an offer, he is made indifferent to the payoff he would receive at the end of line. Thus (10) yields the indifference condition

$$\Phi_{\mathcal{K}_{1p}}(c, \mathcal{N}, \mathcal{K}, \mathcal{A}) = \Phi_{\mathcal{K}_{1p}}\left(p, \mathcal{N}, \mathcal{K}^R, \mathcal{A}\right). \tag{12}$$

According to equation (11), in state $(p, \mathcal{N}, \mathcal{K}, \mathcal{A})$ the equilibrium offer of $\mathcal{K}_{1p}$ makes c indifferent to the payoff he receives in state $(c, \mathcal{N}, \mathcal{K}, \mathcal{A})$, thus (11) yields the indifference condition

$$\Phi_c(p, \mathcal{N}, \mathcal{K}, \mathcal{A}) = \Phi_c(c, \mathcal{N}, \mathcal{K}, \mathcal{A}). \tag{13}$$

Across lists $\mathcal{K}$ and offer types $o \in \{c, p\}$, the indifference conditions (12)–(13) contribute $2 \cdot (\#N(\mathcal{K}^u) - 1)$ equations.

For each offer type o and list $\mathcal{K}$, after $\mathcal{K}_{1p}$ has reached an agreement with c, the game enters a state in which one additional player has an agreement, hence the induction hypothesis prescribes that the payoffs of the players in $N(\mathcal{K}^u - \mathcal{K}_1)$ satisfy the following surplus splitting conditions

$$\Phi_c(o, \mathcal{N}, \mathcal{K}, \mathcal{A}) - \tilde{\Phi}_c(\mathcal{N}^u - ic, \mathcal{N}^u - ic, \emptyset) = \Phi_i(o, \mathcal{N}, \mathcal{K}, \mathcal{A}) - v(i, \mathcal{N}^u - ic)$$

for all $i \in N(\mathcal{K}^u - \mathcal{K}_1) - c$. This contributes $\#N(\mathcal{K}^u) - 2$ equations for each combination of offer type o and list $\mathcal{K}$. Since there are $2 \cdot (\#N(\mathcal{K}^u) - 1)$ such combinations, overall this contributes $2 \cdot (\#N(\mathcal{K}^u) - 1) \cdot (\#N(\mathcal{K}^u) - 2)$ equations.

Finally, for each of the $2 \cdot (\#N(\mathcal{K}^u) - 1)$ combinations of offer type o and list $\mathcal{K}$ we have the adding-up condition

$$\sum_{i \in N(\mathcal{K}^u)} \Phi_i(o, \mathcal{N}, \mathcal{K}, \mathcal{A}) = v\left[\mathcal{N}(\mathcal{K}^u), \mathcal{N}^u\right] - \sum_{(j,T_j) \in \mathcal{A}} T_j.$$

Overall, we have

$$\begin{aligned} &2 \cdot (\#N(\mathcal{K}^u) - 1) + 2 \cdot (\#N(\mathcal{K}^u) - 1) \cdot (\#N(\mathcal{K}^u) - 2) + 2 \cdot (\#N(\mathcal{K}^u) - 1) \\ &2 \cdot (\#N(\mathcal{K}^u) - 1) \#N(\mathcal{K}^u) \end{aligned}$$

equations from indifference, surplus-splitting, and adding-up, respectively, in the same number of unknowns. They are linearly independent [ADD DETAILS] and thus determine a unique solution. It is straightforward to verify that this solution is given by equation (5): simply substitute $\Phi_i(o, \mathcal{N}, \mathcal{K}, \mathcal{K}) = \tilde{\Phi}_i(\mathcal{N}^u, \mathcal{K}^u, \mathcal{A})$ and note that the system reduces exactly to the system of $\#N(\mathcal{K}^u)$ equations (3)–(4).

3 General Graph

The original Rolodex game has a central player that is involved in every negotiation, thus it does not immediately generalize to general graphs. In this section we modify the Rolodex game to accommodate negotiations that are governed by a general graph. Links in this graph negotiate sequentially, with one of the linked players making an offer to the other. If the respondent rejects, the link moves to the end of the line and the direction of the offer is reversed for the next negotiation of this link. As in the original Rolodex game, all agreements are renegotiated in the event of a breakdown. Recall that in the original game a bargaining sessions has two offers, first by the peripheral player and then by the central player, and this sequence of offers is the same in each bargaining session. Here each bargaining session has only one offer, but the direction of the offer is reversed for the next negotiation of the link. Of course the game of this section also works with a star graph. In this case the central planner would again be involved in all negotiations, but with the shortened bargaining sessions with direction reversal. Thus this provides a somewhat different protocol that yields the Myerson-Shapley value for a star graph, yet both protocols have the key feature that a peripheral player rotates to the end of the line after a rejection.

3.1 Extensive Form

We describe the extensive form in a recursive way. At a given point in the game there is a list of directed links $\mathcal{N}$, with at most one link per pair of players. Once again we use the superscript u to obtain the set of undirect links corresponding to a list of directed links. Thus $\mathcal{N}^u$ is the graph that containing the links remaining at this point in the game. There is a second list of directed links $\mathcal{K}$ contains the links that have not yet reached an agreement, in the order in which they will negotiate. Here the direction indicates which player will make an offer when it is the turn of the link to negotiate. There is also a set of agreements $\mathcal{A}$. Each agreement is a pair consisting of a directed link which indicates the direction of the transfer and a transfer amount. The state of the game is described by $(\mathcal{N}, \mathcal{K}, \mathcal{A})$.

The game ends when $\mathcal{K}$ is empty, that is, when all links have reached an agreement. For a player i, let h_i denote the component of $\mathcal{N}^u$ to which the player belongs. Then the payoff of player i is

$$v_i\left(h_i, \mathcal{N}^u\right) - \sum_{\left\{ (jk, T_{jk}) \in \mathcal{A} \middle| j=i \right\}} T_{jk} + \sum_{\left\{ (jk, T_{jk}) \in \mathcal{A} \middle| k=i \right\}} T_{jk}.$$

Here $v_i\left(h_i, \mathcal{N}^u\right)$ is the part of the value $v\left(h_i, \mathcal{N}^u\right)$ that directly goes to player i before any transfers, and it satisfies the adding-up condition

$$\sum_{i \in h} v_i\left(h, \mathcal{N}^u\right) = v\left(h, \mathcal{N}^u\right)$$

for every component $h \in C\left(\mathcal{N}^u\right)$ and every graph $\mathcal{N}^u$. The allocation before transfers given by the functions $v_i\left(h_i, \mathcal{N}^u\right)$ will not matter for equilibrium payoffs, but given this it of course matters for equilibrium transfers.

If $\mathcal{K}$ is nonempty, then the game continues. Let $\mathcal{K}_1$ denote the first directed link in the list $\mathcal{K}$. This link goes from player $\mathcal{K}_{1o}$ to player $\mathcal{K}_{1r}$. The next action is that $\mathcal{K}_{1o}$ offers a transfer $T_{\mathcal{K}_1}$ to player $\mathcal{K}_{1r}$. Player $\mathcal{K}_{1r}$ responds by rejecting or accepting the offer. If $\mathcal{K}_{1r}$ accepts, then link $\mathcal{K}_1$ is deleted from the list of links without agreement and the agreement $(\mathcal{K}_1, T_{\mathcal{K}_1})$ is added to the set $\mathcal{A}$. Thus the game continues in state $[\mathcal{N}, \mathcal{K} - \mathcal{K}_1, \mathcal{A} + (\mathcal{K}_1, T_{\mathcal{K}_1})]$. If $\mathcal{K}_{1r}$ rejects, then with probability $q_{\#\mathcal{K}}$ there is a breakdown. Notice that the breakdown probability is allowed to depend on the number of links without agreement, for reasons to be discussed further below. If there is no breakdown, then link $\mathcal{K}_1$ rotates to the end of the list. Let $\mathcal{K}^R$ denote the resulting new list. The game then continues in state $\left(\mathcal{N}, \mathcal{K}^R, \mathcal{A}\right)$. If there is a breakdown, then the link $\mathcal{K}_1$ is deleted from $\mathcal{N}$ and all agreements are also deleted. The new order of negotiation is given by $\mathcal{N}$ after deletion of $\mathcal{K}_1$. Thus the game continues in state $(\mathcal{N} - \mathcal{K}_1, \mathcal{N} - \mathcal{K}_1, \emptyset)$.

We are interested in the limit as breakdown probabilities approach zero. The role of breakdown is to act as a force towards agreement that ensures uniqueness of no-delay SPE. This force is only needed in situations in which a single link remains without agreement, since otherwise the prospect for a link to rotate to the end of the list already acts as such a force. Thus without loss of generality we set $q_k = 0$ for $k > 1$. Furthermore, notice that for situations with $\#\mathcal{K} = 1$ the extensive form coincides with that of the Binmore-Rubinstein-Wolinsky bilateral alternating offers game with risk of breakdown, which is known to have a unique SPE if the breakdown probability is strictly positive. For simplicity, here we directly focus on the limiting payoffs as the breakdown probability approaches zero.

3.2 No-delay SPE

Let $N(\mathcal{K}^u)$ denote the set of players involved in a link in $\mathcal{K}^u$. For all $i \in N(\mathcal{K}^u)$, recursively define $\tilde{\Phi}_i(\mathcal{N}^u, \mathcal{K}^u, \mathcal{A})$ via the surplus splitting property

$$\tilde{\Phi}_i(\mathcal{N}^u, \mathcal{K}^u, \mathcal{A}) - \tilde{\Phi}_i(\mathcal{N}^u - ij, \mathcal{N}^u - ij, \emptyset) = \tilde{\Phi}_j(\mathcal{N}^u, \mathcal{K}^u, \mathcal{A}) - \tilde{\Phi}_j(\mathcal{N}^u - ij, \mathcal{N}^u - ij, \emptyset) \quad (14)$$

for all $i, j \in N(\mathcal{K}^u)$ with $ij \in \mathcal{K}^u$, and the adding-up property

$$\sum_{i \in h'} \tilde{\Phi}_i(\mathcal{N}^u, \mathcal{K}^u, \mathcal{A}) = v(h, \mathcal{N}^u) + \sum_{\left\{(jk, T_{jk}) \in \mathcal{A} \middle| k \in h'\right\}} T_{jk} - \sum_{\left\{(jk, T_{jk}) \in \mathcal{A} \middle| j \in h'\right\}} T_{jk}. \quad (15)$$

for every $h' \in C(\mathcal{K}^u)$ where h is the component of $\mathcal{N}^u$ that contains h'.

The first sum on the right-hand side is the sum of all already agreed-upon transfers to be received by a set of players that constitutes a component of the set of links $\mathcal{K}^u$ that still have to reach an agreement. The second sum is the sum of all already agreed-upon transfers that are to be paid by this set of players. It follows immediately that

$$\tilde{\Phi}_i(\mathcal{N}^u, \mathcal{N}^u, \emptyset) = \Psi_i(\mathcal{N}^u)$$

since equations (14) and (15) coincide with the properties of fair allocation in equation (1) and component balance in equation (2) for this case, respectively.

We will show that no-delay SPE payoffs in state $(\mathcal{N}, \mathcal{K}, \mathcal{A})$ are unique and given by

$$\Phi_i(\mathcal{N}, \mathcal{K}, \mathcal{A}) = \tilde{\Phi}_i(\mathcal{N}^u, \mathcal{K}^u, \mathcal{A}). \quad (16)$$

Thus the order in which links negotiate does not affect equilibrium payoffs, and in state $(\mathcal{N}, \mathcal{N}, \emptyset)$ in which negotiations start from scratch, equilibrium payoffs are the Myerson-Shapley values for the graph $\mathcal{N}^u$.

We will verify this claim by induction over the number of links that have yet to reach agreement and the number of links remaining in the game. The statement is true for the base case in which there is only one link without agreement, since in that case the payoffs prescribed by equations (14)–(15) coincide with the payoffs in the BRW game. In the induction step, we consider a state $(\mathcal{N}, \mathcal{K}, \mathcal{A})$ with $\#K > 1$ and take as given that the induction hypothesis is true for states in which negotiations start over from scratch with fewer remaining links, and for states in which the number of remaining links in the network is the same but more links have already reached an agreement.

Let $m_i(\mathcal{N},\mathcal{K},\mathcal{A})$ and $M_i(\mathcal{N},\mathcal{K},\mathcal{A})$ denote the infimum and supremum payoff of player i in state $(\mathcal{N},\mathcal{K},\mathcal{A})$ across all no-delay SPEs. Consider state $(\mathcal{N},\mathcal{K},\mathcal{A})$. Without loss of generality, we can restrict attention to lists $\mathcal{K}$ that constitute a component and for which each link is essential for $\mathcal{K}$ to be a component, that is, removing any link would break up $\mathcal{K}$ into multiple components. [ADD MORE DETAIL.] Thus $\#\mathcal{K} = N(\mathcal{K}^u) - 1$.

We will now establish inequalities between the infima and suprema across equilibrium payoffs that together establish uniqueness of no-delay SPE payoffs.

In state $(\mathcal{N},\mathcal{K},\mathcal{A})$ the first link to negotiate is $\mathcal{K}_1$. Here $\mathcal{K}_{1o}$ makes an offer to $\mathcal{K}_{1r}$. If this offer is rejected the state changes to $\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$ and the next link to negotiate is $\mathcal{K}_1^R$, with an offer from $\mathcal{K}_{1o}^R$ to $\mathcal{K}_{1r}^R$. There are two main cases depending on whether a higher transfer from $\mathcal{K}_{1o}^R$ to $\mathcal{K}_{1r}^R$ in state $\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$ is beneficial for the rejection payoff of $\mathcal{K}_{1r}$. Notice that $\mathcal{K}^R$ breaks into two components without the link $\mathcal{K}_1^R$ and the two players $\mathcal{K}_{1o}^R$ and $\mathcal{K}_{1r}^R$ end up in different components. If $\mathcal{K}_{1r}$ ends up in a component with $\mathcal{K}_{1o}^R$, then a higher transfer from $\mathcal{K}_{1o}^R$ is detrimental for $\mathcal{K}_{1r}$. We refer to this as case o since $\mathcal{K}_{1r}$ ends up with the next offer-making player. In case r, $\mathcal{K}_{1r}$ ends up in a component with $\mathcal{K}_{1r}^R$ and thus benefits from a higher transfer from $\mathcal{K}_{1o}^R$ to from $\mathcal{K}_{1r}^R$. In both cases, a lower bound on the payoff of $\mathcal{K}_{1o}^R$ puts an upper bound on the transfer to $\mathcal{K}_{1r}^R$. In case o this puts a lower bound on the rejection payoff of $\mathcal{K}_{1r}$ in state $(\mathcal{N},\mathcal{K},\mathcal{A})$. This in turn puts a upper bound on the payoff of $\mathcal{K}_{1o}$ in state $(\mathcal{N},\mathcal{K},\mathcal{A})$.

We now explicitly derive the relationship between the bounds in case o. Using the induction hypothesis, the payoff of $\mathcal{K}_{1o}^R$ in $\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$ as a function of the transfer T he agrees to make is given by

$$\tilde{\Phi}_{\mathcal{K}_{1o}^R}\left[\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A}+\left(\mathcal{K}_1^R,T\right)\right].$$

Considering this payoff as a function of T, let

$$\tilde{\Phi}^{-1}_{\mathcal{K}_{1o}^R}\left(\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A},\mathcal{K}_1^R,x\right)$$

denote its inverse. A lower bound $m_{\mathcal{K}_{1o}^R}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$ on the payoff of $\mathcal{K}_{1o}^R$ then implies the upper bound

$$T\leq\tilde{\Phi}^{-1}_{\mathcal{K}_{1o}^R}\left[\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A},\mathcal{K}_1^R,m_{\mathcal{K}_{1o}^R}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)\right].$$

Using the induction hypothesis, the payoff of $\mathcal{K}_{1r}$ in state $\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$ as a function of T is

$$\tilde{\Phi}_{\mathcal{K}_{1r}}\left[\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A}+\left(\mathcal{K}_1^R,T\right)\right].$$

In case o this is decreasing in T since $\mathcal{K}_{1r}$ continues in a component with $\mathcal{K}_{1o}^R$. Thus the upper bound on T yields a lower bound

$$\begin{aligned} & m_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right) \\ & \geq \tilde{\Phi}_{\mathcal{K}_{1r}}\left(\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A}+\left\{\mathcal{K}_1^R,\tilde{\Phi}_{\mathcal{K}_{1o}^R}^{-1}\left[\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A},\mathcal{K}_1^R,m_{\mathcal{K}_{1o}^R}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)\right]\right\}\right). \end{aligned} \tag{17}$$

Case r differs in that we obtain an upper bound rather than a lower bound

$$\begin{aligned} & M_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right) \\ & \leq \tilde{\Phi}_{\mathcal{K}_{1r}}\left(\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A}+\left\{\mathcal{K}_1^R,\tilde{\Phi}_{\mathcal{K}_{1o}^R}^{-1}\left[\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A},\mathcal{K}_1^R,m_{\mathcal{K}_{1o}^R}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)\right]\right\}\right). \end{aligned} \tag{18}$$

In inequalities (17)–(18) we have infima on the right hand side. Similar arguments yield the reverse inequalities with infima and suprema swapped, but the same functional form. Thus in case o we have

$$\begin{aligned} & M_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right) \\ & \leq \tilde{\Phi}_{\mathcal{K}_{1r}}\left(\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A}+\left\{\mathcal{K}_1^R,\tilde{\Phi}_{\mathcal{K}_{1o}^R}^{-1}\left[\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A},\mathcal{K}_1^R,M_{\mathcal{K}_{1o}^R}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)\right]\right\}\right) \end{aligned} \tag{19}$$

and in case r we have

$$\begin{aligned} & m_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right) \\ & \geq \tilde{\Phi}_{\mathcal{K}_{1r}}\left(\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A}+\left\{\mathcal{K}_1^R,\tilde{\Phi}_{\mathcal{K}_{1o}^R}^{-1}\left[\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1^R,\mathcal{A},\mathcal{K}_1^R,M_{\mathcal{K}_{1o}^R}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)\right]\right\}\right). \end{aligned} \tag{20}$$

Now consider state $(\mathcal{N},\mathcal{K},\mathcal{A})$. If $\mathcal{K}_{1r}$ accepts an offer T from $\mathcal{K}_{1o}$, then the induction hypothesis implies that his payoff is

$$\tilde{\Phi}_{\mathcal{K}_{1r}}\left[\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1,\mathcal{A}+(\mathcal{K}_1,T)\right].$$

If $\mathcal{K}_{1r}$ rejects, he obtains his payoff in state $\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$. Thus T must be such that $\mathcal{K}_{1r}$ obtains at least this payoff. Hence a lower bound $m_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$ yields a lower bound on T:

$$T\geq\tilde{\Phi}_{\mathcal{K}_{1r}}^{-1}\left[\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1,\mathcal{A},\mathcal{K}_1,m_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)\right]$$

This in turn implies an upper bound on the payoff of $\mathcal{K}_{1o}$:

$$\begin{aligned} & M_{\mathcal{K}_{1o}}\left(\mathcal{N},\mathcal{K},\mathcal{A}\right) \\ & \leq \tilde{\Phi}_{\mathcal{K}_{1o}}\left(\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1,\mathcal{A}+\left\{\mathcal{K}_1,\tilde{\Phi}_{\mathcal{K}_{1r}}^{-1}\left[\mathcal{N},\mathcal{K}^u-\mathcal{K}_1,\mathcal{A},\mathcal{K}_1,m_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)\right]\right\}\right). \end{aligned} \tag{21}$$

Once again we also obtain the flipped inequality

$$\begin{aligned} & m_{\mathcal{K}_{1o}}\left(\mathcal{N},\mathcal{K},\mathcal{A}\right) \\ & \geq \tilde{\Phi}_{\mathcal{K}_{1o}}\left(\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1,\mathcal{A}+\left\{\mathcal{K}_1,\tilde{\Phi}_{\mathcal{K}_{1r}}^{-1}\left[\mathcal{N},\mathcal{K}^u-\mathcal{K}_1,\mathcal{A},\mathcal{K}_1,M_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)\right]\right\}\right). \end{aligned} \tag{22}$$

Having derived inequalities (17)–(22), we can now combine them as follows. We start with inequality (22), which provides a lower bound for $m_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A})$ in terms of $M_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$. Depending on whether case o or r applies, we then use inequality (19) or inequality (18), respectively, to bound $M_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$ in this inequality. This gives us a lower bound for $m_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A})$ in terms of either $M_{\mathcal{K}^R_{1o}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$ or $m_{\mathcal{K}^R_{1o}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$. We keep going like this, cycling through the links until we have a lower bound for $m_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A})$ in terms of $m_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A})$, which we obtain after at most two full cycles. The resulting inequality is

$$m_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A}) \geq \alpha(\mathcal{N},\mathcal{K},\mathcal{A}) + \beta(\mathcal{N},\mathcal{K},\mathcal{A})\, m_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A})$$

for appropriately defined coefficients $\alpha(\mathcal{N},\mathcal{K},\mathcal{A})$ and $\beta(\mathcal{N},\mathcal{K},\mathcal{A})$. It is easy to see that $\beta(\mathcal{N},\mathcal{K},\mathcal{A}) \in (0,1)$, so we obtain

$$m_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A}) \geq \left[1-\beta(\mathcal{N},\mathcal{K},\mathcal{A})\right]^{-1}\alpha(\mathcal{N},\mathcal{K},\mathcal{A}).$$

Proceeding analogously but starting with inequality (21),we obtain

$$M_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A}) \leq \left[1-\beta(\mathcal{N},\mathcal{K},\mathcal{A})\right]^{-1}\alpha(\mathcal{N},\mathcal{K},\mathcal{A}).$$

Thus $m_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A}) = M_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A})$ and (17)–(22) hold as equalities. Thus no-delay SPE payoffs $\Phi_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A})$ and $\Phi_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)$ across the different lists $\mathcal{K}$ are uniquely determined by the system

$$\begin{aligned}
&\Phi_{\mathcal{K}_{1o}}(\mathcal{N},\mathcal{K},\mathcal{A}) \\
&= \tilde{\Phi}_{\mathcal{K}_{1o}}\left(\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}_1,\mathcal{A}+\left\{\mathcal{K}_1,\tilde{\Phi}^{-1}_{\mathcal{K}_{1r}}\left[\mathcal{N},\mathcal{K}^u-\mathcal{K}_1,\mathcal{A},\mathcal{K}_1,\Phi_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)\right]\right\}\right), \qquad (23)\\
&\Phi_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right) \\
&= \tilde{\Phi}_{\mathcal{K}_{1r}}\left(\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}^R_1,\mathcal{A}+\left\{\mathcal{K}^R_1,\tilde{\Phi}^{-1}_{\mathcal{K}^R_{1o}}\left[\mathcal{N}^u,\mathcal{K}^u-\mathcal{K}^R_1,\mathcal{A},\mathcal{K}^R_1,\Phi_{\mathcal{K}^R_{1o}}\left(\mathcal{N},\mathcal{K}^R,\mathcal{A}\right)\right]\right\}\right). \qquad (24)
\end{aligned}$$

[HERE WE HAVE SHOWN UNIQUENESS BUT NOT YET EXISTENCE. ADD BRIEF DISCUSSION OF EXISTENCE, WHICH REQUIRES NON-NEGATIVE GAINS FROM TRADE FOR A LINKS.]

Having established that no-delay SPE payoffs are unique, we now verify that they satisfy the induction hypothesis. We do this by writing down the system of linear equations satisfied by these payoffs and verify that it is satisfied by the payoffs prescribed by the induction hypothesis.

We will solve for $\Phi_i(\mathcal{N},\mathcal{K},\mathcal{A})$ for all $\#N(\mathcal{K}^u)$ players and all the lists of links that the game rotates through starting from an initial list, for given $\mathcal{N}$ and $\mathcal{A}$. Recall that each link in $\mathcal{K}$ is essential for $\mathcal{K}$ to be a single component. This implies that $\mathcal{K}$ contains $\#N(\mathcal{K}^u)-1$ links. Since the direction of links is reversed whenever a link moves to the end of the line, the game moves through $2\left(\#N\left(\mathcal{K}^{u}\right)-1\right)$ different lists until returning to the initial list. Thus we need to solve for $\#N\left(\mathcal{K}^{u}\right)\cdot 2\left(\#N\left(\mathcal{K}^{u}\right)-1\right)$ payoffs.

According to equation (23), in state $(\mathcal{N},\mathcal{K},\mathcal{A})$ player $\mathcal{K}_{1o}$ offers a transfer

$$\tilde{\Phi}_{\mathcal{K}_{1r}}^{-1}\left[\mathcal{N},\mathcal{K}^{u}-\mathcal{K}_{1},\mathcal{A},\mathcal{K}_{1},\Phi_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^{R},\mathcal{A}\right)\right]$$

and this offer is accepted accepted by $\mathcal{K}_{1r}$. By definition this transfer makes $\mathcal{K}_{1r}$ indifferent to the equilibrium payoff $\Phi_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^{R},\mathcal{A}\right)$ he receives in state $\left(\mathcal{N},\mathcal{K}^{R},\mathcal{A}\right)$ in which link $\mathcal{K}_{1}$ is at the end of the line. In other words, when $\mathcal{K}_{1r}$ receives an offer, he is made indifferent to the payoff he would receive if his link with $\mathcal{K}_{1o}$ moves to the end of the line. Thus (23) yields the indifference condition

$$\Phi_{\mathcal{K}_{1r}}(\mathcal{N},\mathcal{K},\mathcal{A})=\Phi_{\mathcal{K}_{1r}}\left(\mathcal{N},\mathcal{K}^{R},\mathcal{A}\right). \tag{25}$$

This contributes $2\cdot\left(\#N\left(\mathcal{K}^{u}\right)-1\right)$ equations, one for each value of $\mathcal{K}$.

For each value of the list $\mathcal{K}$, after link $\mathcal{K}_{1}$ has reached agreement, the game enters a state in which one additional link has reached an agreement and the links without agreement are $\mathcal{K}^{u}-\mathcal{K}_{1}$. The induction hypothesis implies that for a player with a link in $\mathcal{K}^{u}-\mathcal{K}_{1}$ the equilibrium payoff is given by

$$\Phi_{i}(\mathcal{N},\mathcal{K},\mathcal{A})=\tilde{\Phi}_{i}\left[\mathcal{N}^{u},\mathcal{K}^{u}-\mathcal{K}_{1},\mathcal{A}+\left(\mathcal{K}_{1},T_{\mathcal{K}_{1}}\right)\right]$$

and thus for links $ij\in\mathcal{K}^{u}-\mathcal{K}_{1}$ equation (14) implies

$$\Phi_{i}(\mathcal{N},\mathcal{K},\mathcal{A})-\tilde{\Phi}_{i}\left(\mathcal{N}^{u}-ij,\mathcal{N}^{u}-ij,\emptyset\right)=\Phi_{i}(\mathcal{N},\mathcal{K},\mathcal{A})-\tilde{\Phi}_{j}\left(\mathcal{N}^{u}-ij,\mathcal{N}^{u}-ij,\emptyset\right).$$

Since $\mathcal{K}^{u}-\mathcal{K}_{1}$ contains $\#N\left(\mathcal{K}^{u}\right)-2$ links and $\mathcal{K}$ takes $2\cdot\left(\#N\left(\mathcal{K}^{u}\right)-1\right)$ values, this contributes $2\cdot\left(\#N\left(\mathcal{K}^{u}\right)-1\right)\cdot\left(\#N\left(\mathcal{K}^{u}\right)-2\right)$ equations.

Finally, for each value of $\mathcal{K}$ we have the adding-up condition

$$\sum_{i\in N\left(\mathcal{K}^{u}\right)}\Phi_{i}(\mathcal{N},\mathcal{K},\mathcal{A})=v\left(h,\mathcal{N}^{u}\right)+\sum_{\left\{(jk,T_{jk})\in\mathcal{A}\middle|k\in N\left(\mathcal{K}^{u}\right)\right\}}T_{jk}-\sum_{\left\{(jk,T_{jk})\in\mathcal{A}\middle|j\in N\left(\mathcal{K}^{u}\right)\right\}}T_{jk}. \tag{26}$$

where h is the component of $\mathcal{N}^{u}$ containing $\mathcal{K}^{u}$. Since $\mathcal{K}$ takes $2\cdot\left(\#N\left(\mathcal{K}^{u}\right)-1\right)$ values, this is also the number of equations contributed.

Taken together, we have $2\cdot\left(\#N\left(\mathcal{K}^{u}\right)-1\right)\cdot\#N\left(\mathcal{K}^{u}\right)$ equations in the same number of unknowns. These equations are linearly independent [ADD DETAILS] and thus determine a unique solution. It is straightforward to show that this solution is given by (16): simply substitute $\Phi_{i}(\mathcal{N},\mathcal{K},\mathcal{A})=\tilde{\Phi}_{i}\left(\mathcal{N}^{u},\mathcal{K}^{u},\mathcal{A}\right)$ and note that the system reduces exactly to equations (14)–(15).

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