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Block recursive equilibria for stochastic models of search on the job ☆

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Abstract

We develop a general stochastic model of directed search on the job. Directed search allows us to focus on a Block Recursive Equilibrium (BRE) where agents' value functions, policy functions and market tightness do not depend on the distribution of workers over wages and unemployment. We formally prove existence of a BRE under various specifications of workers' preferences and contractual environments, including dynamic contracts and fixed-wage contracts. Solving a BRE is as easy as solving a representative agent model, in contrast to the analytical and computational difficulties in models of random search on the job.

JEL classification: E24; E32; J64

Keywords: Directed search; On-the-job search; Heterogeneity; Aggregate fluctuations

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1. Introduction

This paper studies a general model of search on the job that allows for aggregate shocks, idiosyncratic shocks, and different specifications of the contractual environment. We formally establish existence of a type of equilibria, called block recursive equilibria, which are tractable for studying equilibrium dynamics. To attain this main result, we depart from the bulk of the literature on search on the job, which assumes that search is random or undirected in the sense that a worker does not have any information about the terms of trade offered by different firms before applying for jobs. Instead, we assume that search is directed in the sense that a worker knows the terms of trade offered by different firms before choosing where to apply for a job.[1]

The models of random search on the job by Burdett and Mortensen [5], Postel-Vinay and Robin [23], and Burdett and Coles [4] are a useful tool for studying labor markets because they can simultaneously and parsimoniously explain a number of qualitative features of the data. For example, they can explain the empirical regularities in the transition of workers between employment and unemployment and across jobs that pay different wages (e.g. the negative relationship between job hazard and tenure). They can explain why similar workers employed at similar firms are paid different wages and why wages tend to increase with tenure and experience.[2]

However, these models are difficult to solve outside the steady state because the distribution of workers across different wages and unemployment is an infinite-dimensional state variable which non-trivially affects agents' value and policy functions.[3] This technical feature limits the use of these models. For example, a macroeconomist cannot measure the effect of aggregate productivity shocks on the flows of workers across different employment states and on the wage distribution by simply comparing steady states, unless he has reason to believe that these shocks are very persistent and that the transition phases have negligible length. A public economist cannot measure the welfare effect of a change in the unemployment benefit legislation by comparing two steady states, unless he has reason to believe that agents' discount factor is approximately zero and, hence, the transition phases are unimportant. And if an econometrician estimates the steady state of a model, he has to be careful in using data from a period of time when the fundamentals of the economy have remained approximately unchanged.[4]

Moreover, the hypothesis that the search process is random appears at odds with the empirical evidence. For example, in a recent survey of the US labor market, Hall and Krueger [9,

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[1] The literature of directed search was pioneered by Montgomery [18], Peters [21], Moen [17], Acemoglu and Shimer [1], and Burdett, Shi and Wright [6].

[2] Other popular models of search on the jobs are [19,22], and [3]. These models have qualitative properties that are very different from those of the models by [5,23], and [4]. For example, they cannot generate residual wage inequality.

[3] Recently, Moscarini and Postel-Vinay [20] succeeded in computing the stochastic equilibrium of a model of random search on the job by introducing sufficient firm heterogeneity into Burdett and Mortensen [5]. They are able to solve for the equilibrium because in their model the distribution of workers across employment states varies in a simple way in response to aggregate productivity shocks. However, to obtain this property, they need to assume that the contact probability between firms and workers is exogenous. Therefore, while their model is useful for studying firms' dynamics, it is limited for studying the dynamics of the workers' job-finding rate or the dynamics of unemployment and vacancies.

[4] Postel-Vinay and Robin [23] explicitly acknowledge that estimating the steady state of an OJS model restricts their choice of data: "We have deliberately selected a much shorter period than is available because we want to find out whether it is possible to estimate our model over a homogeneous period of the business cycle. It would have been very hard to defend the assumption of time-invariant parameters (the job offer arrival rate parameters in particular) had we been using a longer panel." Similarly, Jolivet et al. [12] state that they "choose to restrict our analysis to a 3-year sample for three reasons. [...] Third, the model assumes that the labor market is in a steady-state, an assumption that would be harder to defend over a longer period of time."

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Table 1] find that 84 percent of white, male, non-college workers either "knew exactly" or "had a pretty good idea" about how much their current job would pay from the very beginning of the application process (at the time of the first interview). Another piece of evidence against the random search hypothesis and in favor of directed search comes from [11]. Using data from the 1982 Employment Opportunity Pilot Project Survey, this study finds that firms in high-wage industries tend to attract more applicants per vacancy than firms in low-wage industries. These findings should not be surprising, as directed search reflects the fundamental idea in economics that prices help a market allocating resources.

In this paper, we consider a stochastic model of directed search on the job. This model is rather general in that it allows for aggregate and idiosyncratic shocks, and for different specifications of the contractual environment (fixed-wage contracts and dynamic contracts). For this model, we prove existence of an equilibrium in which agents' value and policy functions do not depend on the infinite-dimensional distribution of workers across different employment states. We refer to this equilibrium as a Block Recursive Equilibrium (BRE henceforth). As is accomplished by undirected search models of [5,23], and [4], the BRE of our model generates: (i) worker flows between employment, unemployment, and across employers; (ii) a negative relationship between job hazard and tenure; (iii) residual wage inequality; and (iv) a positive return to tenure and experience. In contrast to these other models, the BRE of our model can be easily computed in and out of the steady state. Therefore, our model can be used, without qualifications, to carry out the labor market measurements that we have described above.

It is precisely the difference in the nature of the search process that explains why our model admits a BRE and the models by Burdett and Mortensen [5], Postel-Vinay and Robin [23], and Burdett and Coles [4] do not. If the search process is directed, workers only apply for jobs that they intend to accept. This self-selection mechanism implies that a firm meets exclusively applicants who are willing to fill its job opening and, hence, its value from a meeting an applicant is independent of the distribution of workers across employment states. This property, together with free entry of firms in the labor market, implies that the probability that a firm meets an applicant and, similarly, the probability that an applicant finds a job are also independent of the distribution of workers across employment states. In turn, the independence of these meeting probabilities implies that the value and policy functions of workers and firms are independent of the distribution of workers across employment states. In contrast, if the search process is random, workers sometimes apply for jobs that they are not willing to accept. Therefore, if the search process is random, the distribution of workers across employment states does affect the probability that the firm meets an applicant that is willing to fill its job opening, the expected value to the firm from meeting an applicant, the equilibrium probability that a firm meets an applicant and, ultimately, the agents' value and policy functions. At the end of Section 5, we will provide a more detailed explanation for why directed search is important for existence of a BRE.

The main contribution of this paper is to prove existence of a BRE for a relatively general model of directed search on the job which allows for aggregate shocks, idiosyncratic shocks, workers' risk aversion, and for different specifications of the contractual environment. By accomplishing this task we intend to provide a solid foundation for future applications of models of directed search on the job. Delacroix and Shi [7] examine a model of directed search on the job with fixed-wage contracts. However, their analysis only focuses on the steady-state equilibrium. Shi [27] was the first to formalize the notion of a BRE and to prove existence of a BRE for a model of directed search on the job. However, his model restricts attention to wage-tenure contracts in a steady state. Menzio and Shi [14] prove existence of a BRE for a stochastic model of directed search on the job and calibrate the model to measure the contribution of aggregate productivity shocks to the cyclical volatility of unemployment, vacancies, and other labor market variables. However, they restrict attention to the case of complete labor contracts. In order to generalize the results from Shi [27] and Menzio and Shi [14], the current paper has to develop a different existence proof. For example, the existence proof in Menzio and Shi [14] is based on the equivalence between the solution to the social planner's problem and the equilibrium allocation, which does not hold when employment contracts are incomplete.

2. The model

2.1. Agents and markets

The economy is populated by a continuum of infinitely-lived workers with measure one and a continuum of firms with positive measure. Each worker has a periodical utility function $\upsilon(.)$ defined over consumption, where $\upsilon : \mathbb{R} \to \mathbb{R}$ is a twice-continuously differentiable, strictly increasing, weakly concave function such that $\upsilon'(.) \in [\underline{\upsilon}', \bar{\upsilon}']$, $0 < \underline{\upsilon}' \leqslant \bar{\upsilon}'$. Each worker maximizes the expected sum of periodical utilities discounted at the factor $\beta \in (0, 1)$. The unemployment benefit is b.

Each firm operates a technology with constant returns to scale which turns one unit of labor into $y + z$ units of consumption. The first component of productivity, y, is common to all firms, and its value lies in the set $Y = \{y_1, y_2, \ldots, y_{N(y)}\}$, where $\underline{y} \equiv y_1 < \cdots < y_{N(y)} \equiv \bar{y}$ and $N(y) \geqslant 2$ is an integer. The second component of productivity, z, is specific to each firm-worker pair, and its value lies in the set $Z = \{z_1, z_2, \ldots, z_{N(z)}\}$, where $\underline{z} \equiv z_1 < \cdots < z_{N(z)} \equiv \bar{z}$ and $N(z) \geqslant 1$ is an integer. Each firm maximizes the expected sum of periodical profits discounted at the factor β.

The labor market is organized in a continuum of submarkets indexed by the expected lifetime utility x that the firms offer to the workers, $x \in X = [\underline{x}, \bar{x}]$, with $\underline{x} < \upsilon(b)/(1 - \beta)$ and $\bar{x} > \upsilon(\bar{y} + \bar{z})/(1 - \beta)$. Specifically, whenever a firm meets a worker in submarket x, the firm offers the worker an employment contract that gives him the expected lifetime utility x. In submarket x, the ratio of the number of vacancies created by firms to the number of workers looking for jobs is given by the tightness $\theta(x, \psi) \geqslant 0$ and is determined in the equilibrium, where ψ is the aggregate state of the economy described below.[5]

Time is discrete and continues forever. At the beginning of each period, the state of the economy can be summarized by the triple $(y, u, g) \equiv \psi$. The first element of ψ denotes the aggregate component of labor productivity, $y \in Y$. The second element denotes the measure of workers who are unemployed, $u \in [0, 1]$. The third element is a function $g : X \times Z \to [0, 1]$, with $g(V, z)$ denoting the measure of workers who are employed at jobs that give them the lifetime utility $\tilde{V} \leqslant V$ and that have an idiosyncratic component of productivity $\tilde{z} \leqslant z$.

Each period is divided into four stages: separation, search, matching and production. During the separation stage, an employed worker is forced to move into unemployment with probability $\delta \in (0, 1)$. Also, during the separation stage, an employed worker has the option to voluntarily move into unemployment.

During the second stage, a worker gets the opportunity to search for a job with a probability that depends on his recent employment history. In particular, if the worker was unemployed at

[5] In submarkets that are not visited by any workers, $\theta(x, \psi)$ is an out-of-equilibrium conjecture that helps determining the equilibrium behavior.

the beginning of the period, he can send an application with probability $\lambda_u \in (0,1]$. If the worker was employed at the beginning of the period and did not lose his job during the separation stage, he can search with probability $\lambda_e \in (0,1]$. If the worker lost his job during the separation stage, he cannot search immediately. Conditional on being able to search, the worker chooses which submarket to visit. In this sense, search is directed. Also, during the search stage, a firm chooses how many vacancies to create and where to locate them. The cost of maintaining a vacancy for one period is $k > 0$. Both workers and firms take the tightness $\theta(x,\psi)$ parametrically.[6]

During the matching stage, the workers and the vacancies in submarket x come together through a frictional meeting process. In particular, a worker meets a vacant job with probability $p(\theta(x,\psi))$, where $p:\mathbb{R}_+ \to [0,1]$ is a twice-continuously differentiable, strictly increasing, strictly concave function such that $p(0)=0$ and $p'(0)<\infty$. Similarly, a vacancy meets a worker with probability $q(\theta(x,\psi))$, where $q:\mathbb{R}_+ \to [0,1]$ is a twice-continuously differentiable, strictly decreasing, convex function such that $q(\theta)=p(\theta)/\theta$, $q(0)=1$, $q'(0)<0$, and $p(q^{-1}(.))$ concave.[7] When a vacancy and a worker meet, the firm that owns the vacancy offers to the worker an employment contract that gives him the lifetime utility x. If the worker rejects the offer, he returns to his previous employment position. If the worker accepts the offer, the two parties form a new match. To simplify the exposition, we assume that all new matches have the idiosyncratic component of productivity $z_0 \in Z$.

During the last stage, an unemployed worker produces and consumes $b \in (0, \bar{y}+\bar{z})$ units of output. A worker employed at a job z produces $y+z$ units of output and consumes w of them, where w is specified by the worker's labor contract.[8] At the end of the production stage, Nature draws next period's aggregate component of productivity, $\hat{y}$, from the probability distribution $\Phi_{\hat{y}}(\hat{y}|y)$, and next period's idiosyncratic component of productivity, $\hat{z}$, from the probability distribution $\Phi_{\hat{z}}(\hat{z}|z)$.[9] The draws of the idiosyncratic component of productivity are independent across matches.[10]

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6 In this paper, workers choose which submarket to visit and firms choose where to locate their vacancies, given that the tightness in each submarket x is described by $\theta(x,\psi)$. This search-and-matching process generates the same equilibrium conditions as the more naturalistic model in which firms post employment contracts for their vacancies and workers choose where to apply for a job (e.g. [1]).

7 The assumption on $p(q^{-1}(.))$ is needed to guarantee that the worker's search problem is strictly concave and, hence, has a unique solution. This assumption is satisfied by some common specifications such as the urn-ball matching function, $q_0(\theta)=1-e^{-1/\theta}$, and the generalized form of the telephone-line matching function, $q_0(\theta)=(\frac{\alpha}{\alpha+\theta^\gamma})^{1/\gamma}$, where $\gamma \geqslant 1$ and $\alpha \in (0,1]$. Modify these functions as $q(\theta)=(1-\varepsilon)q_0(\theta)+\varepsilon/(1+\theta)$, where $\varepsilon \in (0,1)$ can be an arbitrarily small number. Then, the modified functions satisfy all of the assumptions that we have imposed and, especially, the assumption that $q'(\theta)<0$ for all $\theta \geqslant 0$. Note that these assumptions are sufficient, but not necessary, for a BRE to exist.

8 Part of the assumption of timing is that employed workers can voluntarily move into unemployment only at the beginning of the period. This assumption is made entirely for easing exposition and it is not important for the analysis, since our proof of existence can be easily modified to allow employed workers to move into unemployment at the beginning of the production stage. Although the assumption allows for the possibility that some workers at the production stage might be employed at jobs that give them a lifetime utility of $V<U$, the possibility does not arise when the model is calibrated to the US economy (see Section 7).

9 Throughout this paper, the caret on a variable indicates the variable in the next period.

10 To ease exposition, we restrict attention to aggregate and idiosyncratic shocks that affect only labor productivity. However, the existence proof of a BRE does not depend on this choice, and can be easily generalized to the case in which aggregate and idiosyncratic shocks affect the search process, the value of unemployment, labor income taxes, etc.

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2.2. Contractual environment

We consider two alternative contractual environments. In the first environment, the firm commits to an employment contract that specifies the worker's wage as a function of the history of realizations of the idiosyncratic productivity of the match, z, the history of realizations of the aggregate state of the economy, ψ, and the history of realizations of a two-point lottery that is drawn at the beginning of every production stage.[11] In the remainder of the paper, we shall refer to this environment as the one with "dynamic contracts", since we will formulate the contracts recursively as in the literature on dynamic contracts (e.g., [2]).[12] In the second environment, the firm commits to a wage that remains constant throughout the entire duration of the employment relationship. This constant wage is allowed to depend only on the outcome of a two-point lottery that is drawn at the beginning of the employment relationship. In the remainder of the paper, we shall refer to this environment as the one with "fixed-wage contracts".

We are interested in these two contractual environments because they have been the focus of the literature on random search on the job. The "dynamic contract" environment generalizes the environment considered by Burdett and Coles [4] and Shi [27] to an economy with stochastic productivity.[13] The "fixed-wage contract" environment has been considered by Burdett and Mortensen [5] and [12]. Notice that, in both environments, contracts are incomplete because wages cannot be made contingent upon the outside offers received by the worker.

2.3. Worker's problem

Consider a worker whose current job gives him a lifetime utility V and who has the opportunity to look for a job at the beginning of the search stage. His search decision is to choose which submarket x to visit. If the worker visits submarket x, he succeeds in finding a job with probability $p(\theta(x, \psi))$, and fails with probability $1 - p(\theta(x, \psi))$. If he succeeds, he enters the production stage in a new employment relationship which gives him the lifetime utility x. If he fails to find a new match (or if he does not apply for a job), he enters the production stage by retaining his current employment position, which gives him a lifetime utility V. Therefore, the worker's lifetime utility at the beginning of the search stage is $V + \max\{0, R(V, \psi)\}$, where R is the search value function (i.e., the return to search) defined as

$$R(V, \psi) = \max_{x \in X} p\big(\theta(x, \psi)\big)(x - V). \tag{2.1}$$

Denote $m(V, \psi)$ as the solution to the maximization problem in (2.1), and $\tilde{p}(V, \psi)$ as the composite function $p(\theta(m(V, \psi), \psi))$.

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[11] We allow for the lottery in order to guarantee that the profit of the firm is a concave function of the value of the employment contract to the worker. In this sense, lotteries play a similar role in our model as in [24]. In turn, concavity of the profit of the firm (together with concavity of the composite function $p(q^{-1}(.))$) guarantees that the search problem of the worker is strictly concave in the choice x and so its solution is unique (see the proof of Lemma 4.1). Finally, the uniqueness of the search strategy of the worker is needed to establish the continuity of the equilibrium mapping T (see the proof of Lemma 5.2).

[12] In contrast to most models in the literature on dynamic contracts, however, there is no private information in our model, and a worker can quit for another contract or into unemployment in any period during the contract.

[13] In the special case where workers are risk neutral, the dynamic contracts considered in this paper attain the same allocation as the complete contracts considered in Menzio and Shi [14] do. Therefore, the proof of existence of a BRE in this paper generalizes the existence proof in Menzio and Shi [14].

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Next, consider an unemployed worker at the beginning of the production stage, and denote as $U(\psi)$ his lifetime utility. In the current period, the worker produces and consumes b units of output. During the next search stage period, the worker is unemployed and has the opportunity to look for a job with probability λ_u. Therefore, the worker's lifetime utility $U(\psi)$ is equal to

$$U(\psi) = \upsilon(b) + \beta \mathbb{E}_{\hat{\psi}}\big[U(\hat{\psi}) + \lambda_u \max\big\{0, R\big(U(\hat{\psi}), \hat{\psi}\big)\big\}\big]. \tag{2.2}$$

2.4. Firm's problem

2.4.1. Dynamic contracts

Consider a firm that has just met a worker in submarket x. The firm offers to the worker an employment contract that specifies his wage at every future date as a function of the realized history of the idiosyncratic productivity of the match, the realized history of the aggregate state of the economy, and the history of realizations of a two-point lottery that is drawn at the beginning of every production stage. The firm chooses the contract to maximize its profits while delivering the promised lifetime utility x to the worker. Characterizing the solution to this problem is difficult because the dimension of the history upon which wages are contingent grows to infinity with time. However, following the literature on dynamic contracts (e.g. [2]), we can rewrite this problem recursively by using the worker's lifetime utility as an auxiliary state variable.[14]

In the recursive formulation of the problem, the state of the contract at the beginning of the production stage in an arbitrary period is described by the worker's lifetime utility, V, the state of the aggregate economy, ψ, and the idiosyncratic productivity of the match, z. (If the period is the first period of the contract, then $V = x$.) Let s denote (ψ, z). Given V and s, the firm chooses a two-point lottery over the worker's wage w in the current period, the worker's probability d of becoming unemployed in the next separation stage, and the worker's lifetime utility $\hat{V}$ at the beginning of the next production stage. That is, the firm chooses a two-point lottery $c = (\pi_i, w_i, d_i, \hat{V}_i)_{i=1}^2$, where π_i is the probability that the realization of the lottery is $(w_i, d_i, \hat{V}_i)$. Note that d_i and $\hat{V}_i$ are plans contingent on $\hat{s}$ because they will be realized in the next period. The firm chooses c to maximize the sum of its profits from the current period onward. Therefore, the firm's maximized value $J(V, s)$ is equal to

$$\begin{aligned}
J(V, s) = {} & \max_{\pi_i, w_i, \hat{d}_i, \hat{V}_i} \sum_{i=1}^{2} \pi_i \big\{ y + z - w_i \\
& + \beta \mathbb{E}_{\hat{s}}\big[\big(1 - d_i(\hat{s})\big)\big(1 - \lambda_e \tilde{p}\big(\hat{V}_i(\hat{s}), \hat{\psi}\big)\big) J\big(\hat{V}_i(\hat{s}), \hat{s}\big)\big]\big\}, \\
\text{s.t.} \quad & \pi_i \in [0, 1], \quad w_i \in \mathbb{R}, \quad d_i : \Psi \times Z \to [\delta, 1], \quad \hat{V}_i : \Psi \times Z \to X, \quad \text{for } i = 1, 2, \\
& \sum_{i=1}^{2} \pi_i = 1, \qquad d_i(\hat{s}) = \big\{\delta \text{ if } U(\hat{\psi}) \leqslant \hat{V}_i(\hat{s}) + \lambda_e R\big(\hat{V}_i(\hat{s}), \hat{\psi}\big),\ 1 \text{ else}\big\}, \\
& \sum_{i=1}^{2} \pi_i \big\{\upsilon(w_i) + \beta \mathbb{E}_{\hat{s}}\big[d_i(\hat{s}) U(\hat{\psi}) + \big(1 - d_i(\hat{s})\big)\big(\hat{V}_i(\hat{s}) + \lambda_e R\big(\hat{V}_i(\hat{s}), \hat{\psi}\big)\big)\big]\big\} = V.
\end{aligned} \tag{2.3}$$

[14] More precisely, we can prove that the value function of the firm's contracting problem is the unique solution to the recursive problem (2.3). Also, we can prove that the firm's contracting problem yields the same solutions as the recursive problem (2.3). The proofs of these equivalence results are standard and available upon request.

The last constraint is the promise-keeping constraint, which requires c to provide the worker with the lifetime utility V. The second last constraint is the individual rationality constraint on separation, which requires the separation probability d to be consistent with the worker's incentives to quit into unemployment. We denote the optimal policy function associated with (2.3) as $c(V,s) = (\pi_i, w_i, d_i, \hat{V}_i)_{i=1}^2$, where $\pi_i = \pi_i(V,s)$, $w_i = w_i(V,s)$, $d_i = d_i(V,s,\hat{s})$, and $\hat{V}_i = \hat{V}_i(V,s,\hat{s})$, for $i = 1,2$.

2.4.2. Fixed-wage contracts

With fixed-wage contracts, we assume that workers are risk neutral; i.e., $\upsilon(w) = w$ for all w. Consider a worker who is employed for a wage of w at the beginning of the production stage, and denote as $H(w,\psi)$ his lifetime utility. In the current period, the worker consumes w units of output. During the next separation stage, the worker is forced by Nature to become unemployed with probability δ, and has the option of keeping his job with probability $1-\delta$. If the worker becomes unemployed, he does not have the opportunity to look for a new job during the next search stage. If the workers keeps his job, he has the opportunity to look for a better job with probability λ_e. Therefore, the worker's lifetime utility $H(w,\psi)$ is equal to

$$\begin{aligned} H(w,\psi) &= w + \beta \mathbb{E}_{\hat{\psi}} \big\{ d(\hat{\psi}) U(\hat{\psi}) + \big(1 - d(\hat{\psi})\big)\big[H(w,\hat{\psi}) \\ &\qquad + \lambda_e \max\big\{0, R\big(H(w,\hat{\psi}),\hat{\psi}\big)\big\}\big]\big\}, \\ d(\hat{\psi}) &= \big\{\delta \text{ if } U(\hat{\psi}) \leqslant H(w,\hat{\psi}) + \lambda_e \max\big\{0, R\big(H(w,\hat{\psi}),\hat{\psi}\big)\big\},\ 1 \text{ else}\big\}. \end{aligned} \tag{2.4}$$

We denote as $h(V,\psi)$ the wage that provides an employed worker with the lifetime utility V. That is, $h(V,\psi)$ is the solution for w to the equation $H(w,\psi) = V$.

Next, consider a firm that employs a worker for a wage of w at the beginning of the production stage, and denote as $K(w,s)$ its lifetime profit. In the current period, the firm's profit is given by $y + z - w$. The discounted sum of profits from the next period onward is $(1 - d(\hat{\psi}))[1 - \lambda_e \tilde{p}(H(w,\hat{\psi}),\hat{\psi})]K(w,\hat{s})$. Therefore, $K(w,s)$ is equal to

$$\begin{aligned} K(w,s) &= y + z - w + \beta \mathbb{E}_{\hat{s}} \big\{\big(1 - d(\hat{\psi})\big)\big[1 - \lambda_e \tilde{p}\big(H(w,\hat{\psi}),\hat{\psi}\big)\big]K(w,\hat{s})\big\}, \\ d(\hat{\psi}) &= \big\{\delta \text{ if } U(\hat{\psi}) \leqslant H(w,\hat{\psi}) + \lambda_e \max\big\{0, R\big(H(w,\hat{\psi}),\hat{\psi}\big)\big\},\ 1 \text{ else}\big\}. \end{aligned} \tag{2.5}$$

Finally, consider a firm that has just met a worker in submarket $x = V$, and denote as $J(V,\psi,z_0)$ its lifetime profit. The firm offers to the worker a two-point lottery over the constant wage w. The firm's offer is required to provide the worker with the lifetime utility V (if accepted). Therefore, the firm's lifetime profit $J(V,\psi,z_0)$ is equal to

$$\begin{aligned} J(V,\psi,z_0) &= \max_{\pi_i,\tilde{V}_i} \sum_{i=1}^{2} \pi_i K\big(h(\tilde{V}_i,\psi),\psi,z_0\big), \\ \text{s.t.}\quad & \pi_i \in [0,1], \quad \tilde{V}_i \in X, \quad \text{for } i = 1,2, \\ & \sum_{i=1}^{2} \pi_i = 1, \qquad \sum_{i=1}^{2} \pi_i \tilde{V}_i = V. \end{aligned} \tag{2.6}$$

We denote the optimal policy function associated with (2.6) as $c = (\pi_i, \tilde{V}_i)_{i=1}^2$, where $\pi_i = \pi_i(V,s)$ and $\tilde{V}_i = \tilde{V}_i(V,s)$, for $i = 1,2$.

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2.5. Market tightness

During the search stage, a firm chooses how many vacancies to create and where to locate them. The firm's benefit of creating a vacancy in submarket x is the product between the matching probability, $q(\theta(x,\psi))$, and the value of meeting a worker, $J(x,\psi,z_0)$. The firm's cost of creating a vacancy is k. When the benefit is strictly smaller than the cost, the firm's optimal policy is to create no vacancies in x. When the benefit is strictly greater than the cost, the firm's optimal policy is to create infinitely many vacancies in x. And when the benefit and the cost are equal, the firm's profit is independent of the number of vacancies it creates in submarket x.

In any submarket that is visited by a positive number of workers, the tightness $\theta(x;y)$ is consistent with the firm's optimal creation strategy if and only if

$$k \geqslant q\big(\theta(x,\psi)\big)J(x,\psi,z_0), \tag{2.7}$$

and $\theta(x,\psi)\geqslant 0$, with complementary slackness. In any submarket that workers do not visit, the tightness $\theta(x,\psi)$ is consistent with the firm's optimal creation strategy if and only if $q(\theta(x,\psi))J(x,\psi,z_0)$ is smaller or equal than k. However, following the rest of the literature on directed search on the job (i.e. Shi [27] and Menzio and Shi [14]), we restrict attention to equilibria in which the tightness $\theta(x,\psi)$ satisfies the above complementary slackness condition in every submarket.[15]

3. BRE: definition and procedure

The previous section motivates the following definition of a recursive equilibrium:

Definition 3.1. A *Recursive Equilibrium* consists of a market tightness function $\theta : X\times\Psi\to\mathbb{R}_+$, a search value function $R : X\times\Psi\to\mathbb{R}$, a search policy function $m : X\times\Psi\to X$, an unemployment value function $U:\Psi\to\mathbb{R}$, a firm's value function $J : X\times\Psi\times Z\to\mathbb{R}$, a contract policy function $c : X\times\Psi\times Z\to C$, and a transition probability function for the aggregate state of the economy $\Phi_{\hat\psi} : \Psi\times\Psi\to[0,1]$. These functions satisfy the following requirements:

(i) θ satisfies (2.7) for all $(x,\psi)\in X\times\Psi$;
(ii) R satisfies (2.1) for all $(V,\psi)\in X\times\Psi$, and m is the associated policy function;
(iii) U satisfies (2.2) for all $\psi\in\Psi$;
(iv) J satisfies (2.3) or (2.6) for all $(V,\psi,z)\in X\times\Psi\times Z$, and c is the associated policy function;
(v) $\Phi_{\hat\psi}$ is derived from the policy functions, (m,c), and the probability distributions for $(\hat y,\hat z)$.

Solving a recursive equilibrium outside the steady state requires solving a system of functional equations in which the unknown functions depend on the entire distribution of workers across employment states, (u,g). Since the dimension of this distribution is large (and infinite in the version of the model with dynamic contracts), solving a recursive equilibrium outside the steady state is a difficult task both analytically and computationally. In contrast, solving the following

[15] The literature on directed search off the job (e.g. [1,13,17]) assumes that, in a submarket that is not active in equilibrium, the tightness is such that a worker would be indifferent between visiting that submarket and the submarket that he visits in equilibrium. In models with directed search on the job, workers are heterogeneous ex post and, hence, it is more convenient to use the firm's indifference condition (2.7) to pin down the tightness of inactive submarkets.

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class of equilibria is much easier because it involves solving a system of functional equations in which the unknown functions have at most three dimensions.

Definition 3.2. A BRE (Block Recursive Equilibrium) is a recursive equilibrium such that the functions $\{\theta, R, m, U, J, c\}$ depend on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y, and not through the distribution of workers across employment states, (u, g).

In this paper, we establish existence of a BRE. To this aim, we define $\mathcal{J}(X \times Y \times Z)$ (henceforth $\mathcal{J}$) as the set of firms' value functions $J : X \times Y \times Z \to \mathbb{R}$ such that: (J1) For all $(y, z) \in Y \times Z$ and all $V_1, V_2 \in X$, with $V_1 \leqslant V_2$, the difference $J(V_2, y, z) - J(V_1, y, z)$ is bounded between $-\bar{B}_J(V_2 - V_1)$ and $-\underline{B}_J(V_2 - V_1)$, where $\bar{B}_J \geqslant \underline{B}_J > 0$ are constants; (J2) For all $(V, y, z) \in X \times Y \times Z$, $J(V, y, z)$ is bounded in $[\underline{J}, \bar{J}]$[16]; (J3) For all $(y, z) \in Y \times Z$, $J(V, y, z)$ is concave in V. In words, a function J in the set $\mathcal{J}$ depends on ψ only through y; it is bounded, and strictly decreasing and weakly concave in V; and its "derivative" with respect to V is bounded above and below, i.e. J is bi-Lipschitz continuous in V.[17] In Appendix A, we prove that $\mathcal{J}$ is a non-empty, bounded, closed and convex subset of the space of bounded, continuous functions on $X \times Y \times Z$, with the sup norm.[18]

In Section 4, we take an arbitrary firm's value function J from the set $\mathcal{J}$. Given J, we prove that the market tightness function, θ, that solves the equilibrium condition (2.7) depends on the state of the economy, ψ, only through the aggregate component of productivity, y, and not through the distribution of workers across employment states, (u, g). Intuitively, since the value of filling a vacancy in submarket x does not depend on the distribution of workers and the cost of creating a vacancy is constant, the equilibrium probability of filling a vacancy in submarket x, and hence the tightness of submarket x, must be independent of the distribution of workers.

Given θ, we prove that the search value function, R, that solves the equilibrium condition (2.1) depends on ψ only through y. Intuitively, R does not depend on (u, g), because neither the probability that a worker finds a job in submarket x nor the benefit to a worker from finding a job in submarket x depends on the employment status of other workers in the economy. Given R, we prove that the unemployment value function, U, that solves the equilibrium condition (2.2) depends on ψ only through y. Intuitively, U does not depend on (u, g), because neither the output of an unemployed worker nor his return to search depends on the distribution of workers across different employment states.

In Section 5, we insert J, θ, R, and U in the RHS of the equilibrium condition (2.3) to construct an update of the firm's value function, where T maps the function J with which the above procedure starts into a new function. First, we prove that TJ depends on ψ only through y. Intuitively, TJ does not depend on (u, g) because the output of a match in the current period, the probability that a match survives until the next production stage, and the value to the firm of a

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[16] We list this property separately in addition to (J1) to emphasize the fact that the bounds $\underline{J}$ and $\bar{J}$ are uniform for all functions in the set $\mathcal{J}$ and for all (V, y, z).

[17] A function $J(x)$ is Lipschitz over $x \in X$ if $|J(x_2) - J(x_1)| \leqslant B_1|x_2 - x_1|$ for all $x_1, x_2 \in X$, where B_1 is a finite constant. The function is bi-Lipschitz if, in addition, $|J(x_2) - J(x_1)| \geqslant B_2|x_2 - x_1|$ for all $x_1, x_2 \in X$, where B_2 is a strictly positive constant. We need the firm's value function J to be bi-Lipschitz in order to ensure the set $\mathcal{J}$ to be closed and convex. In addition, bi-Lipschitz continuity implies that J is strictly decreasing, a property that will be used to establish important properties such as those of the market tightness.

[18] Throughout this paper, the norm is the sup norm unless it is specified otherwise.

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match at the next production stage are all independent of the distribution of workers across employment states. Second, we prove that TJ is bounded between $\underline{J}$ and $\bar{J}$; it is strictly decreasing and weakly concave in V; and its "derivative" with respect to V is bounded between $-\bar{B}_J$ and $-\underline{B}_J$. Intuitively, the firm's updated value function, TJ, is bounded because the output of the match is bounded and there is time discounting; TJ is decreasing because a firm finds it costly to provide a worker with higher lifetime utility; TJ is concave because the contract between a firm and a worker includes a lottery; and the "derivative" of TJ is bounded because the derivative of the worker's utility function is bounded. Third, we prove that TJ is continuous in J.

From the first two properties of TJ above, it follows that the equilibrium operator T maps the set of firm's value functions $\mathcal{J}$ into itself. From the third property of TJ, it follows that the equilibrium operator T is continuous in J. From bi-Lipschitz continuity of TJ, it follows that the family of functions $T(\mathcal{J})$ is equicontinuous. Overall, the equilibrium operator T satisfies the assumptions of Schauder's fixed point theorem (see [28, Theorem 17.4]), and, hence, there exists a $J^* \in \mathcal{J}$ such that $J^* = TJ^*$. Applying one more time the above procedure that leads to the mapping T, but with the firm's value function J^*, we can construct equilibrium policy functions θ^*, R^*, m^*, U^*, and c^*. These functions and J^* constitute a BRE for the version of the model with dynamic contracts. In Section 6, we use a similar argument to prove existence of a BRE for the version of the model with fixed-wage contracts.

4. General properties of an equilibrium

In this section we take an arbitrary $J \in \mathcal{J}$ as a firm's value function. Given J, we compute the market tightness function, θ, the search value and policy functions, R and m, and the value function of unemployment, U, that solve the equilibrium conditions (2.7), (2.1) and (2.2). Then, we prove that these functions depend on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y, and not through the distribution of workers across employment states, (u, g). Clearly, this property is necessary to establish existence of a BRE. Next, we characterize the functions θ, R and m. In particular, we prove that the market tightness function, θ, is Lipschitz continuous and decreasing in x, that the search value function, R, is Lipschitz continuous and decreasing in V, and that the search policy function, m, is Lipschitz continuous and increasing in V. Finally, we prove that the functions θ, R, m and U are continuous in the firm's value function J with which they are computed. These properties will be used in Section 5 to establish that the equilibrium operator T is a continuous mapping on J.

4.1. Market tightness

Starting with an arbitrary value function of the firm, $J \in \mathcal{J}$, we construct the market tightness function and analyze its properties. For all $(x, \psi) \in X \times \Psi$ such that $J(x, y, z_0) \geqslant k$, the solution to the equilibrium condition (2.7) is given by a market tightness $q^{-1}(k/J(x, y, z_0))$, where $q^{-1}(k/J(x, y, z_0))$ is bounded between 0 and $\bar{\theta} \equiv q^{-1}(k/\bar{J})$. For all $(x, \psi) \in X \times \Psi$ such that $J(x, y, z_0) < k$, the solution to the equilibrium condition (2.7) is given by a market tightness 0. The condition $J(x, y, z_0) \geqslant k$ is satisfied if and only if $x \leqslant \tilde{x}(y)$, where $\tilde{x}(y)$ is the solution to the equation $J(x, y, z_0) = k$ with respect to x. From these observations, it follows that the function $\theta : X \times Y \to [0, \bar{\theta}]$ defined as

$$\theta(x, y) = \begin{cases} q^{-1}(k/J(x, y, z_0)), & \text{if } x \leqslant \tilde{x}(y), \\ 0, & \text{else}, \end{cases} \tag{4.1}$$

is the unique solution to the equilibrium condition (2.7) for all $(x, \psi) \in X \times \Psi$.

The market tightness function, θ, has several properties. First, θ depends on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y, and not through the distribution of workers across different employment states, (u, g). Second, the market tightness function, θ, is strictly decreasing with respect to x. Intuitively, since the firm's value from filling a vacancy is lower in a submarket with a higher x, the firm's probability of filling a vacancy must be higher. Third, the market tightness function, θ, is Lipschitz continuous in x for all x, and bi-Lipschitz in x for $x < \tilde{x}(y)$. Intuitively, since the firm's value function, J, is bi-Lipschitz continuous in x and the derivative of the function $q^{-1}(.)$ is bounded, the market tightness function defined in (4.1) is also bi-Lipschitz continuous for all such x that $\theta(x, y) > 0$. Finally, the probability that a worker meets a vacancy in submarket x, $p(\theta(x, y))$, decreases at an increasing rate as x increases. This property follows from the concavity of the firm's value function J and of the composite function $p(q^{-1}(.))$. These properties of θ are summarized in the following lemma and proved in Appendix B.

Lemma 4.1.

(i) *For all $y \in Y$, the market tightness function, θ, is such that*

$$\frac{\bar{B}_J}{q'(\bar{\theta})k}(x_2 - x_1) \leqslant \theta(x_2, y) - \theta(x_1, y) \leqslant \frac{\underline{B}_J k}{q'(0)\bar{J}^2}(x_2 - x_1), \quad \text{if } x_1 \leqslant x_2 \leqslant \tilde{x}(y),$$
$$\frac{\bar{B}_J}{q'(\bar{\theta})k}(x_2 - x_1) \leqslant \theta(x_2, y) - \theta(x_1, y) \leqslant 0, \quad \text{if } x_1 \leqslant \tilde{x}(y) \leqslant x_2,$$
$$\theta(x_2, y) - \theta(x_1, y) = 0, \quad \text{if } \tilde{x}(y) \leqslant x_1 \leqslant x_2, \tag{4.2}$$

where $\underline{B}_J$ and $\bar{B}_J$ are the bi-Lipschitz bounds on all functions in $\mathcal{J}$.

(ii) *For all $y \in Y$ and all $x \in [\underline{x}, \tilde{x}(y)]$, the composite function $p(\theta(x, y))$ is strictly decreasing and strictly concave in x.*

The function $\theta(x, y)$ constructed above depends on the arbitrary function J. Consider two arbitrary functions $J_n, J_r \in \mathcal{J}$. Let θ_n denote the market tightness function computed with J_n, and θ_r with J_r. In the following lemma, we prove that, if the distance between J_n and J_r converges to zero, so does the distance between θ_n and θ_r. That is, the market tightness function, θ, is continuous in the firm's value function J with which it is computed.

Lemma 4.2. *For any $\rho > 0$ and any $J_n, J_r \in \mathcal{J}$, if $\|J_n - J_r\| < \rho$, then*

$$\|\theta_n - \theta_r\| < \alpha_\theta \rho, \quad \alpha_\theta \equiv -\bar{B}_J / \left[q'(\bar{\theta})\underline{B}_J k\right]. \tag{4.3}$$

Proof. For the sake of brevity, let us suppress the dependence of various functions on (y, z). Let $\rho > 0$ be an arbitrary real number. Let J_n and J_r be arbitrary functions in $\mathcal{J}$ such that $\|J_n - J_r\| < \rho$. Let y be an arbitrary point in Y. From property (J1) of the set $\mathcal{J}$, it follows that $J_r(x + \underline{B}_J^{-1}\rho) - J_r(x) \leqslant -\rho$ and, hence, $J_r(x) - \rho \geqslant J_r(x + \underline{B}_J^{-1}\rho)$. From property (J1), it also follows that $J_r(x) - J_r(x - \underline{B}_J^{-1}\rho) \leqslant -\rho$ and, hence, $J_r(x) + \rho \leqslant J_r(x - \underline{B}_J^{-1}\rho)$. From these observations and $\|J_n - J_r\| < \rho$, it follows that

$$J_n(x) < J_r(x) + \rho \leqslant J_r\left(x - \underline{B}_J^{-1}\rho\right),$$
$$J_n(x) > J_r(x) - \rho \geqslant J_r\left(x + \underline{B}_J^{-1}\rho\right). \tag{4.4}$$

From the first line in (4.4) and Eq. (4.1), it follows that $\theta_n(x) \leqslant \theta_r(x - \underline{B}_J^{-1}\rho)$. Similarly, from the second line in (4.4) and Eq. (4.1), it follows that $\theta_n(x) \geqslant \theta_r(x + \underline{B}_J^{-1}\rho)$. Hence,

$$\theta_n(x) - \theta_r(x) < \theta_r\big(x - \underline{B}_J^{-1}\rho\big) - \theta_r(x) \leqslant \alpha_\theta \rho,$$
$$\theta_n(x) - \theta_r(x) > \theta_r\big(x + \underline{B}_J^{-1}\rho\big) - \theta_r(x) \geqslant -\alpha_\theta \rho,$$

where α_θ is defined in (4.3). Thus, $|\theta_n(x) - \theta_r(x)| \leqslant \alpha_\theta \rho$. Since this result holds for all $(x, y, z) \in X \times Y \times Z$, we conclude that $\|\theta_n - \theta_r\| < \alpha_\theta \rho$. □

4.2. *Search problem*

Given the firm's value function $J \in \mathcal{J}$, the market tightness function θ defined in (4.1) satisfies the equilibrium condition (2.7). Given θ, the search value function, R, that satisfies the equilibrium condition (2.1) is equal to $\max_{x \in X} f(x, V, y)$ for all $(x, \psi) \in X \times \Psi$, where $f(x, V, y) \equiv p(\theta(x, y))(x - V)$. Note that, for all $(V, \psi) \in X \times \Psi$, the objective function, f, depends on the aggregate state of the economy, ψ, through the aggregate component of productivity, y, and not through the distribution of workers across different employment states, (u, g). Also, note that the choice set, X, is independent of the aggregate state of the economy, ψ. From these observations, it follows that the optimal search decision and the search value function, R, depend on ψ only through y and not through (u, g).

Given θ, a search policy function satisfies the equilibrium condition (2.1) if its value belongs to $\arg\max_{x \in X} f(x, V, y)$ for all $(V, \psi) \in X \times \Psi$. For all $(V, \psi) \in X \times \Psi$, the objective function, f, is negative for all x in the interval $[\underline{x}, V]$, strictly positive for all x in the interval $(V, \tilde{x}(y))$, and equal to zero for all x in the interval $[\tilde{x}(y), \bar{x}]$. Moreover, the objective function is strictly concave in x for all x in the interval $(V, \tilde{x}(y))$ (see the proof of Lemma 3.1 in Shi [27]). Therefore, if $V < \tilde{x}(y)$, the arg max is unique and belongs to the interval $(V, \tilde{x}(y))$. If $V \geqslant \tilde{x}(y)$, the arg max includes any point between V and $\bar{x}$. From these observations, it follows that the unique solution to (2.1) is the function $m : X \times Y \to X$ defined as

$$m(V, y) = \begin{cases} \arg\max_{x \in X} f(x, V, y), & \text{if } V < \tilde{x}(y), \\ V, & \text{else.} \end{cases} \tag{4.5}$$

In Lemma 4.3, we prove that the return to search, R, is decreasing in V. Intuitively, since the value to a worker from finding a job in submarket x is decreasing in the value of his current employment position, V, and the probability that a worker finds a job in submarket x is independent of V, the return to search is decreasing in V. Also, in Lemma 4.3, we prove that the search policy function, m, is increasing in V. Intuitively, since the marginal rate of substitution between the value offered by a new job and the probability of finding a new job is decreasing in V, the optimal search strategy is increasing in V.

Lemma 4.3. *For all $y \in Y$ and all $V_1, V_2 \in X$, $V_1 \leqslant V_2$, the search value function, R, satisfies*:

$$-(V_2 - V_1) \leqslant R(V_2, y) - R(V_1, y) \leqslant 0, \tag{4.6}$$

and the search policy function, m, is such that

$$0 \leqslant m(V_2, y) - m(V_1, y) \leqslant V_2 - V_1. \tag{4.7}$$

Proof. For the sake of brevity, let us suppress the dependence of the functions θ, $\tilde{x}$, m and p on y. Let V_1 and V_2 be two arbitrary points in X, with $V_1 \leqslant V_2$. We have:

$$\begin{aligned}R(V_2)-R(V_1)&\leqslant f\big(m(V_2),V_2\big)-f\big(m(V_2),V_1\big)\leqslant -p\big(\theta\big(m(V_2)\big)\big)(V_2-V_1)\leqslant 0,\\ R(V_2)-R(V_1)&\geqslant f\big(m(V_1),V_2\big)-f\big(m(V_1),V_1\big)\geqslant -p\big(\theta\big(m(V_1)\big)\big)(V_2-V_1)\\ &\geqslant -(V_2-V_1),\end{aligned}$$

where the first inequality in both lines uses the fact that $R(V_i)$ is equal to $f(m(V_i),V_i)$ and greater than $f(m(V_{-i}),V_i)$ where $-i\neq i$ and $i,-i=1,2$. Thus, (4.6) holds.

Turn to (4.7). If $V_1\geqslant\tilde{x}$, then $m(V_2)=V_2$ and $m(V_1)=V_1$. In this case, (4.7) clearly holds. If $V_2\geqslant\tilde{x}\geqslant V_1$, then $m(V_2)=V_2$ and $m(V_1)\in(V_1,\tilde{x})$. Also in this case, (4.7) holds.

Now, consider the remaining case where $V_1\leqslant V_2<\tilde{x}$. Since $f(m(V_1),V_1)\geqslant f(m(V_2),V_1)$ and $f(m(V_2),V_2)\geqslant f(m(V_1),V_2)$, we have

$$\begin{aligned}0&\geqslant f\big(m(V_2),V_1\big)-f\big(m(V_1),V_1\big)+f\big(m(V_1),V_2\big)-f\big(m(V_2),V_2\big)\\ &=p\big(\theta\big(m(V_2)\big)\big)(V_2-V_1)-p\big(\theta\big(m(V_1)\big)\big)(V_2-V_1)\\ &=\big[p\big(\theta\big(m(V_2)\big)\big)-p\big(\theta\big(m(V_1)\big)\big)\big](V_2-V_1).\end{aligned}$$

Since $p(\theta(x))$ is decreasing in x, the previous inequality implies that $m(V_2)\geqslant m(V_1)$.

If $m(V_2)=m(V_1)$, (4.7) holds. If $m(V_2)>m(V_1)$, let Δ be an arbitrary real number in the open interval between 0 and $(m(V_2)-m(V_1))/2$. Using the definition of R, we can deduce from the inequality $f(m(V_1),V_1)\geqslant f(m(V_1)+\Delta,V_1)$ the following result:

$$m(V_1)-V_1\geqslant\frac{p(\theta(m(V_1)+\Delta))\Delta}{p(\theta(m(V_1)))-p(\theta(m(V_1)+\Delta))}.$$

Similarly, because $f(m(V_2),V_2)\geqslant f(m(V_2)-\Delta,V_2)$, we have

$$m(V_2)-V_2\leqslant\frac{p(\theta(m(V_2)-\Delta))\Delta}{p(\theta(m(V_2)-\Delta))-p(\theta(m(V_2)))}.$$

Recall that the function $p(\theta(x))$ is decreasing and concave in x for all $x\leqslant\tilde{x}(y)$. Since $m(V_1)+\Delta\leqslant m(V_2)-\Delta$, then $p(\theta(m(V_1)+\Delta))\geqslant p(\theta(m(V_2)-\Delta))$. Similarly, since $m(V_1)<m(V_2)$, $p(\theta(m(V_1)))-p(\theta(m(V_1)+\Delta))\leqslant p(\theta(m(V_2)-\Delta))-p(\theta(m(V_2)))$. From these observations and the inequalities above, it follows that $m(V_2)-m(V_1)\leqslant V_2-V_1$. Hence, (4.7) holds. □

Now we turn to the composite function $\tilde{p}(V,y)=p(\theta(m(V,y),y))$. The function $\tilde{p}(V,y)$ is the probability that an employed worker finds a new job during the matching stage, given that his current job gives him the lifetime utility V and the aggregate productivity is y. The following corollary states that the function $\tilde{p}(V,y)$ is decreasing and Lipschitz continuous in V:

Corollary 4.4. *For all $y\in Y$ and all $V_1,V_2\in X$, $V_1\leqslant V_2$, the quitting probability $\tilde{p}$ is such that*

$$-\bar{B}_p(V_2-V_1)\leqslant\tilde{p}(V_2,y)-\tilde{p}(V_1,y)\leqslant-\underline{B}_p(V_2-V_1),\tag{4.8}$$

where $\bar{B}_p=-p'(0)\bar{B}_J/[q'(\bar{\theta})k]>0$ and $\underline{B}_p=0$.

Proof. Let y be an arbitrary point in Y, and let V_1, V_2 be two points in X with $V_1\leqslant V_2$. From Lemma 4.3, it follows that the difference $m(V_2,y)-m(V_1,y)$ is greater than 0 and smaller than V_2-V_1. From Lemma 4.1, it follows that the difference $\theta(m(V_2,y),y)-\theta(m(V_2,y),y)$ is greater than $(V_2-V_1)\bar{B}_J/[q'(\bar{\theta})k]$ and smaller than 0. Finally, since p is a concave function of θ, the difference $p(\theta(m(V_2,y),y))-p(\theta(m(V_2,y),y))$ is such that

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$$\frac{p'(0)\bar{B}_J}{q'(\bar{\theta})k}(V_2 - V_1) \leqslant p\big(\theta\big(m(V_2, y), y\big)\big) - p\big(\theta\big(m(V_1, y), y\big)\big) \leqslant 0.$$

These are the bounds in (4.8). □

Now, consider two arbitrary functions $J_n, J_r \in \mathcal{J}$. Let θ_n denote the market tightness function computed with J_n, R_n and m_n the search value and policy functions computed with θ_n, and $\tilde{p}_n(V, y)$ the composite function $p(\theta_n(m_n(V, y), y))$. Similarly, let θ_r, R_r, m_r and $\tilde{p}_r(V, y)$ be the functions computed with J_r. In the following lemma, we first prove that, if the distance between J_n and J_r converges to zero, so do the distances between R_n and R_r and between $\tilde{p}_n$ and $\tilde{p}_r$. While it is intuitive that R is continuous in J, proving continuity of $\tilde{p}$ in J is more involved, because $\tilde{p}$ depends on the policy function m. In principle, it may be possible that the policy functions m_n and m_r are far apart even when the value functions R_n and R_r are close to each other. To prove continuity of $\tilde{p}$ in J, we explore concavity of the composite function $p(\theta(x))$.

Lemma 4.5. *For any $\rho > 0$ and any $J_n, J_r \in \mathcal{J}$, if $\|J_n - J_r\| < \rho$, then*

$$\|R_n - R_r\| < \alpha_R \rho, \quad \alpha_R \equiv p'(0)\alpha_\theta(\bar{x} - \underline{x}), \tag{4.9}$$

$$\|\tilde{p}_n - \tilde{p}_r\| < \alpha_p(\rho), \quad \alpha_p(\rho) \equiv \max\big\{2\bar{B}_p\rho^{1/2} + p'(0)\alpha_\theta\rho, 2\alpha_R\rho^{1/2}\big\}. \tag{4.10}$$

As $\rho \to 0$, $\alpha_p(\rho) \to 0$.

Proof. For the sake of brevity, let us suppress the dependence of various functions on V and y. Let $\rho > 0$ be an arbitrary real number. Let J_n and J_r be arbitrary functions in $\mathcal{J}$ such that $\|J_n - J_r\| < \rho$. Let (V, y) be an arbitrary point in $X \times Y$. We have:

$$\begin{aligned}
|R_n - R_r| &\leqslant \max_{x\in X}\big|\big[p\big(\theta_n(x)\big) - p\big(\theta_r(x)\big)\big](x - V)\big| \\
&\leqslant \Big\{\max_{x\in X}\big|p\big(\theta_n(x)\big) - p\big(\theta_r(x)\big)\big|\Big\}\Big\{\max_{x\in X}|x - V|\Big\} \\
&\leqslant \left\{\max_{x\in X}\left|\int_{\theta_r(x)}^{\theta_n(x)} p'(t)\,dt\right|\right\}(\bar{x} - \underline{x}) < p'(0)\alpha_\theta(\bar{x} - \underline{x})\rho,
\end{aligned}$$

where the last inequality uses the bounds in (4.3). Since this result holds for all $(V, y) \in X \times Y$, we conclude that $\|R_n - R_r\| < \alpha_R\rho$.

Now, consider the function $\tilde{p}$. Without loss of generality, assume $m_r(V, y) \leqslant m_n(V, y)$. (If $m_r(V, y) > m_n(V, y)$, just switch the roles of m_n and m_r in the proof below.) First, consider the case where $p(\theta_r(m_r)) \leqslant p(\theta_n(m_n))$. In this case, we have:

$$(0 \leqslant)\ p\big(\theta_n(m_n)\big) - p\big(\theta_r(m_r)\big) \leqslant p\big(\theta_n(m_r)\big) - p\big(\theta_r(m_r)\big) < p'(0)\alpha_\theta\rho,$$

where the first inequality uses the fact that $p(\theta_n(x))$ is decreasing in x and $m_n \geqslant m_r$, and the second inequality uses the bounds in (4.3).

Second, consider the case where $p(\theta_r(m_r)) > p(\theta_n(m_n))$ and $m_n - 2\rho^{1/2} \leqslant m_r \leqslant m_n$. In this case, the distance between $p(\theta_n(m_n))$ and $p(\theta_r(m_r))$ is such that

$$\begin{aligned}
(0 <)\ p\big(\theta_r(m_r)\big) - p\big(\theta_n(m_n)\big) &= p\big(\theta_r(m_r)\big) - p\big(\theta_r(m_n)\big) + p\big(\theta_r(m_n)\big) - p\big(\theta_n(m_n)\big) \\
&< 2\bar{B}_p\rho^{1/2} + p'(0)\alpha_\theta\rho,
\end{aligned}$$

where the last inequality uses the bounds in (4.8) and in (4.3). Note that this bound is larger than the one in the previous case.

Finally, consider the remaining case where $p(\theta_r(m_r)) > p(\theta_n(m_n))$ and $m_r < m_n - 2\rho^{1/2} < m_n$. First, note that $m_r \geqslant V$, because $m_r \in (V, \tilde{x}_r)$ if $V < \tilde{x}_r$, and $m_r = V$ if $V \geqslant \tilde{x}_r$. This observation implies that $m_n > V + \rho^{1/2}$, because if $m_n \leqslant V + \rho^{1/2}$ then $m_r < V - \rho^{1/2} < V$, which is a contradiction. Second, note that if $V < \tilde{x}_n$, then $m_n \in (V, \tilde{x}_n)$ and, if $V \geqslant \tilde{x}_n$, then $m_n = V$. Since $m_n > V$, this observation implies that $m_n < \tilde{x}_n$.

Note that $p(\theta_n(m_n))(m_n - V) \geqslant p(\theta_n(m_n - \rho^{1/2}))(m_n - \rho^{1/2} - V)$, because m_n is the optimal search decision when $J = J_n$. Therefore, we have

$$\begin{aligned} p\big(\theta_n(m_n)\big)\rho^{1/2} &\geqslant \big[p\big(\theta_n\big(m_n - \rho^{1/2}\big)\big) - p\big(\theta_n(m_n)\big)\big]\big(m_n - \rho^{1/2} - V\big) \\ &\geqslant \big[p\big(\theta_n(m_r)\big) - p\big(\theta_n\big(m_r + \rho^{1/2}\big)\big)\big]\big(m_n - \rho^{1/2} - V\big) \\ &\geqslant \big[p\big(\theta_n(m_r)\big) - p\big(\theta_n\big(m_r + \rho^{1/2}\big)\big)\big]\big(m_r + \rho^{1/2} - V\big). \end{aligned}$$

To obtain the second inequality we have used the facts that $p(\theta_n(x))$ is concave in x for all $x \in [\underline{x}, \tilde{x}_n]$, that $m_r + \rho^{1/2} < m_n < \tilde{x}_n$, and that $m_n - \rho^{1/2} - V > 0$. To obtain the third inequality we have used the facts that $m_r + \rho^{1/2} < m_n - \rho^{1/2}$, and that $p(\theta_n(m_r)) > p(\theta_n(m_r + \rho^{1/2}))$. Next, note that $p(\theta_r(m_r))(m_r - V)$ is greater than $p(\theta_r(m_r + \rho^{1/2}))(m_r + \rho^{1/2} - V)$. Therefore, we have

$$p\big(\theta_r(m_r)\big)\rho^{1/2} \leqslant \big[p\big(\theta_r(m_r)\big) - p\big(\theta_r\big(m_r + \rho^{1/2}\big)\big)\big]\big(m_r + \rho^{1/2} - V\big).$$

Subtracting this inequality from the previous result and dividing by $\rho^{1/2}$, we obtain

$$\begin{aligned} (0 <)\ p\big(\theta_r(m_r)\big) - p\big(\theta_n(m_n)\big) &\leqslant \rho^{-1/2}\big[p\big(\theta_r(m_r)\big) - p\big(\theta_n(m_r)\big) + p\big(\theta_n\big(m_r + \rho^{1/2}\big)\big) \\ &\quad - p\big(\theta_r\big(m_r + \rho^{1/2}\big)\big)\big]\big(m_r + \rho^{1/2} - V\big) \\ &< 2p'(0)\alpha_\theta \rho^{1/2}(\bar{x} - \underline{x}) = 2\alpha_R \rho, \end{aligned}$$

where the last line uses the fact that the distance between $p(\theta_r(m))$ and $p(\theta_n(m))$ is smaller than $p'(0)\alpha_\theta \rho$, and that $m_r + \rho^{1/2} - V$ is smaller than $\bar{x} - \underline{x}$.

Overall, we have established that the distance between $p(\theta_r(m_r))$ and $p(\theta_n(m_n))$ is such that

$$\big|p\big(\theta_r(m_r)\big) - p\big(\theta_n(m_n)\big)\big| < \max\big\{2\bar{B}_p\rho^{1/2} + p'(0)\alpha_\theta\rho, 2\alpha_R\rho^{1/2}\big\} = \alpha_p(\rho).$$

Since this result holds for all $(V, y) \in X \times Y$, we conclude that $\|\tilde{p}_r - \tilde{p}_n\| < \alpha_p(\rho)$. □

4.3. Unemployment value

Given the firm's value function $J \in \mathcal{J}$, the solution to the equilibrium condition (2.7) is the market tightness, θ, defined in (4.1). Given θ, the solution to the equilibrium condition (2.1) is the search value function, R, defined as $R(V, y) = \max_{x \in X} f(x, V, y)$. Given R, the value function of unemployment is a solution to the equilibrium condition (2.2) if and only if it is a fixed point of the mapping T_U defined as

$$(T_U\varphi)(\psi) = \upsilon(b) + \beta\mathbb{E}_{\hat{\psi}}\big\{\varphi(\hat{\psi}) + \lambda_u \max\big\{0, R\big(\varphi(\hat{\psi}), \hat{y}\big)\big\}\big\}. \tag{4.11}$$

In the next lemma, we prove that the mapping T_U has a unique fixed point in the set $\mathcal{C}(Y)$ of bounded continuous functions $\varphi : Y \to \mathbb{R}$. Therefore, there exists a unique value function of unemployment, $U \in \mathcal{C}(Y)$, that satisfies the equilibrium condition (2.2), and that depends on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y, but not through the distribution of workers across different employment states, (u, g).

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Lemma 4.6.

(i) *There exists a unique function* $U \in \mathcal{C}(Y)$ *such that* $U = T_U U$.
(ii) *For all* $y \in Y$, $U(y) \in [\underline{U}, \bar{U}]$, *where* $\underline{U} = (1-\beta)^{-1}\upsilon(b) > \underline{x}$ *and* $\bar{U} = \upsilon(b) + \beta\bar{x} < \bar{x}$.

Proof. In Appendix C. □

Now, consider two arbitrary functions $J_n, J_r \in \mathcal{J}$. Let θ_n denote the market tightness function computed with J_n, R_n the search value function computed with θ_n, and U_n the unemployment value function computed with R_n. Similarly, let θ_r, R_r and U_r be the functions generated from J_r. In the following lemma, we prove that, if the distance between J_n and J_r converges to zero, so does the distance between U_n and U_r.

Lemma 4.7. *For any* $\rho > 0$ *and any* $J_n, J_r \in \mathcal{J}$, *if* $\|J_n - J_r\| < \rho$, *then*

$$\|U_n - U_r\| < \alpha_U \rho, \quad \alpha_U \equiv \beta\lambda_u\alpha_R/(1-\beta). \tag{4.12}$$

Proof. For the sake of brevity, let us suppress the dependence of various functions on $\hat{y}$. Let $\rho > 0$ be an arbitrary real number. Let J_n and J_r be arbitrary functions in $\mathcal{J}$ such that $\|J_n - J_r\| < \rho$. Let y be an arbitrary point in Y. The distance between $U_n(y)$ and $U_r(y)$ is such that

$$\begin{aligned}
&\left|U_n(y) - U_r(y)\right| \\
&\quad \leqslant \beta\mathbb{E}_{\hat{y}}\left\{\left|\left[U_n + \lambda_u R_n(U_n)\right] - \left[U_r + \lambda_u \max R_n(U_r)\right]\right| + \lambda_u\left|R_n(U_r) - R_r(U_r)\right|\right\} \\
&\quad < \beta\|U_n - U_r\| + \beta\lambda_u\alpha_R\rho.
\end{aligned}$$

To obtain the second inequality we have used the fact that the distance between $U_n + \lambda_u R_n(U_n)$ and $U_n + \lambda_u R_n(U_n)$ is smaller than the distance between U_n and U_r. Since the above result holds for all $y \in Y$, it follows that $\|U_n - U_r\| < \alpha_u\rho$. □

5. BRE with dynamic contracts

In the previous section, we took an arbitrary firm's value function, $J \in \mathcal{J}$. Given J, we computed the market tightness, θ, the search value and policy functions, R and m, and the unemployment value function, U, that solve the equilibrium conditions (2.7), (2.1) and (2.2). In this section, we insert J, θ, R, m and U into the RHS of the equilibrium condition (2.3) to compute an update, $\tilde{J}$, of the firm's value function J. This process implicitly defines an operator T through $\tilde{J} = TJ$. In Section 5.1, we prove that $\tilde{J}$ depends on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y, and not through the distribution of workers across employment states, (u, g). Then, we prove that $\tilde{J}$ satisfies the properties (J1), (J2) and (J3) of the set $\mathcal{J}$. These findings imply that the operator T maps the set $\mathcal{J}$ into itself. In Section 5.1, we use the properties of θ, R, m, and U in order to prove that the operator T is continuous in J. Finally, in Section 5.2, we use Schauder's fixed point theorem to prove that the operator T has a fixed point and, hence, that a BRE exists.

5.1. Updated value function of the firm

Consider an arbitrary firm's value function $J \in \mathcal{J}$. Let θ denote the market tightness function that satisfies the equilibrium condition (2.7) given J. Let R and m denote the search value and

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policy functions that satisfy the equilibrium condition (2.1) given θ. Let U denote the unemployment value function that satisfies the equilibrium condition (2.2) given R. Inserting J, θ, R, m and U into the RHS of the equilibrium condition (2.3), we obtain an update, $\tilde{J}$, of the firm's value function J. More specifically, $\tilde{J}$ is given by[19]

$$\begin{aligned}
\tilde{J}(V,y,z) = \max_{\pi_i, w_i, d_i, \hat{V}_i} \sum_{i=1}^{2} \pi_i \big\{ & y + z - w_i \\
& + \beta \mathbb{E}_{\hat{s}} \big[\big(1 - d_i(\hat{y},\hat{z})\big)\big(1 - \lambda_e \tilde{p}\big(\hat{V}_i(\hat{y},\hat{z}),\hat{y}\big)\big) J\big(\hat{V}_i(\hat{y},\hat{z}),\hat{y},\hat{z}\big)\big]\big\}, \\
\text{s.t.} \quad & \pi_i \in [0,1], \quad w_i \in \mathbb{R}, \quad d_i : Y \times Z \to [\delta,1], \quad \hat{V}_i : Y \times Z \to X, \quad \text{for } i = 1,2, \\
& \sum_{i=1}^{2} \pi_i = 1, \qquad d_i(\hat{y},\hat{z}) = \big\{\delta \text{ if } U(\hat{y}) \leqslant \hat{V}_i(\hat{y},\hat{z}) + \lambda_e R\big(\hat{V}_i(\hat{y},\hat{z}),\hat{y}\big),\ 1 \text{ else}\big\}, \\
& \sum_{i=1}^{2} \pi_i \big\{ \upsilon(w_i) + \beta \mathbb{E}_{\hat{s}}\big[d_i(\hat{y},\hat{z}) U(\hat{y}) + \big(1 - d_i(\hat{y},\hat{z})\big)\big(\hat{V}_i(\hat{y},\hat{z}) \\
& \quad + \lambda_e R\big(\hat{V}_i(\hat{y},\hat{z}),\hat{y}\big)\big)\big]\big\} = V.
\end{aligned} \tag{5.1}$$

The updated value function of the firm, $\tilde{J}$, has four important properties. First, $\tilde{J}$ depends on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y, and not through the distribution of workers across different employment states, (u,g). This property follows immediately from the fact that both the objective function and the choice set on the RHS of (5.1) depend on y but not on (u,g). Second, the updated value function, $\tilde{J}$, is bi-Lipschitz continuous in V. More specifically, for all $(y,z) \in Y \times Z$ and all $V_1, V_2 \in X$, with $V_1 \leqslant V_2$, the difference $\tilde{J}(V_2,y,z) - \tilde{J}(V_1,y,z)$ is bounded between $-(V_2 - V_1)/\underline{\upsilon}'$ and $-(V_2 - V_1)/\bar{\upsilon}'$ (see part (i) in the proof of Lemma 5.1). Third, $\tilde{J}$ is bounded in $[\underline{J}, \bar{J}]$, where the bounds $\underline{J}$ and $\bar{J}$ are independent of J and $\tilde{J}$ (see (5.2) below). Finally, $\tilde{J}$ is concave in V, as a result of the use of the lottery in the contract (see part (iii) in the proof of Lemma 5.1).

The bounds $\underline{B}_J$, $\bar{B}_J$, $\underline{J}$, and $\bar{J}$ are set as

$$\begin{aligned}
& \bar{B}_J = \frac{1}{\underline{\upsilon}'}, \qquad \underline{B}_J = \frac{1}{\bar{\upsilon}'}, \\
& -\underline{J} = \bar{J} = \max\left\{ \frac{|\bar{y} + \bar{z} - \upsilon^{-1}(\underline{x} - \beta\bar{x})|}{1 - \beta(1-\delta)}, \frac{|\underline{y} + \underline{z} - \upsilon^{-1}(\bar{x} - \beta\underline{x})|}{1 - \beta(1-\delta)} \right\}.
\end{aligned} \tag{5.2}$$

With these bounds, $\tilde{J}$ satisfies conditions (J1)–(J3) and, hence, belongs to the set $\mathcal{J}$, as stated in the next lemma.

Lemma 5.1. *Set the bounds $\underline{B}_J$, $\bar{B}_J$, $\underline{J}$, and $\bar{J}$ as in* (5.2)*. Then, the updated value function, $\tilde{J}$, belongs to the set $\mathcal{J}$.*

[19] In a BRE, the distribution of workers across employment states in the next period, $(\hat{u},\hat{g})$, is uniquely determined by the realization of the aggregate component of productivity in the next period, $\hat{y}$, and by the state of the economy in the current period, ψ. Therefore, in the contracting problem (5.1), next period's separation probability, d_i, and continuation value, $\hat{V}_i$, can be written as functions of $\hat{y}$ and z only.

Proof. For all $(V, y, z) \in X \times Y \times Z$, $\tilde{J}(V, y, z)$ is equal to $\max_{\gamma \in \Gamma} F(\gamma, V, y, z)$, where γ is defined as the tuple $(\pi_1, \tilde{V}_1, \hat{V}_1, \hat{V}_2)$; Γ is defined as the set of γ's such that $\pi_1 \in [0, 1)$, $\tilde{V}_1 \in \mathbb{R}$, $\hat{V}_1 : Y \times Z \to X$, and $\hat{V}_2 : Y \times Z \to X$; and $F(\gamma, V, y, z)$ is defined as

$$\begin{aligned} F(\gamma, V, y, z) = \sum_{i=1}^{2} \pi_i \bigl\{ & y + z - w_i \\ & + \beta \mathbb{E}_{\hat{s}} \bigl[\bigl(1 - d_i(\hat{y}, \hat{z})\bigr)\bigl(1 - \lambda_e \tilde{p}\bigl(\hat{V}_i(\hat{y}, \hat{z}), \hat{y}\bigr)\bigr) J\bigl(\hat{V}_i(\hat{y}, \hat{z}), \hat{y}, \hat{z}\bigr)\bigr]\bigr\}, \\ \text{s.t.} \quad & \pi_2 = 1 - \pi_1, \qquad \tilde{V}_2 = (V - \pi_1 \tilde{V}_1)/\pi_2, \\ & d_i(\hat{y}, \hat{z}) = \bigl\{ \delta \text{ if } U(\hat{y}) \leqslant \hat{V}_i(\hat{y}, \hat{z}) + \lambda_e R\bigl(\hat{V}_i(\hat{y}, \hat{z}), \hat{y}\bigr),\ 1 \text{ else} \bigr\}, \\ & w_i = \upsilon^{-1}\bigl(\tilde{V}_i - \beta \mathbb{E}_{\hat{s}}\bigl[d_i(\hat{y}, \hat{z}) U(\hat{y}) + \bigl(1 - d_i(\hat{y}, \hat{z})\bigr)\bigl(\hat{V}_i(\hat{y}, \hat{z}) + \lambda_e R\bigl(\hat{V}_i(\hat{y}, \hat{z}), \hat{y}\bigr)\bigr)\bigr]\bigr). \end{aligned} \tag{5.3}$$

Denote $F'(\gamma, V, y, z)$ as the derivative of $F(\gamma, V, y, z)$ with respect to V. It is easy to verify that

$$F'(\gamma, V, y, z) = -\frac{1}{\upsilon'(w_2)} \in \left[-\frac{1}{\underline{\upsilon}'}, -\frac{1}{\bar{\upsilon}'} \right].$$

(i) First, we want to prove that $\tilde{J}$ satisfies property (J1) of the set $\mathcal{J}$. To this aim, let (y, z) be an arbitrary point in $Y \times Z$, and let V_1, V_2 be two points in X with $V_1 \leqslant V_2$. The distance between $\tilde{J}(V_2, y, z)$ and $\tilde{J}(V_1, y, z)$ is such that

$$\begin{aligned} \bigl|\tilde{J}(V_2, y, z) - \tilde{J}(V_1, y, z)\bigr| &\leqslant \max_{\gamma \in \Gamma} \bigl| F(\gamma, V_2, y, z) - F(\gamma, V_1, y, z) \bigr| \\ &\leqslant \max_{\gamma \in \Gamma} \left| \int_{V_1}^{V_2} F'(\gamma, t, y, z)\, dt \right| \\ &\leqslant \max_{\gamma \in \Gamma} \int_{V_1}^{V_2} \bigl| F'(\gamma, t, y, z) \bigr|\, dt \leqslant |V_2 - V_1| / \underline{\upsilon}'. \end{aligned}$$

The inequality above implies that the function $\tilde{J}$ is Lipschitz continuous in V. Therefore, it is absolutely continuous and almost everywhere differentiable with respect to V (see [25, p. 112]). The function F is differentiable with respect to V. Therefore, at any point of differentiability, the derivative of $\tilde{J}$ with respect to V is equal to $F'(\gamma^*(V, y, z), V, y, z)$, where $\gamma^*(V, y, z)$ belongs to $\arg\max_{\gamma \in \Gamma} F(\gamma, V, y, z)$ (see [16, Theorem 1]). From these properties of $\tilde{J}$, it follows that the difference $\tilde{J}(V_2, y, z) - \tilde{J}(V_1, y, z)$ is such that

$$\tilde{J}(V_2, y, z) - \tilde{J}(V_1, y, z) = \int_{V_1}^{V_2} F'\bigl(\gamma^*(t, y, z), t, y, z\bigr)\, dt \in \left[-\frac{V_2 - V_1}{\underline{\upsilon}'}, -\frac{V_2 - V_1}{\bar{\upsilon}'} \right].$$

(ii) Next, we want to prove that $\tilde{J}$ satisfies property (J2) of the set $\mathcal{J}$. To this aim, let (V, y, z) be an arbitrary point in $X \times Y \times Z$. Also, let γ_0 denote the tuple $(\pi_{1,0}, \tilde{V}_{1,0}, \hat{V}_{1,0}, \hat{V}_{2,0})$, where $\pi_{1,0} = 0$, $\tilde{V}_{1,0} = \bar{x}$, $\hat{V}_{1,0} = \hat{V}_{2,0} = \underline{x}$. The firm's value $\tilde{J}(V, y, z)$ is such that

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$$\tilde{J}(V,y,z) \leqslant \bar{y}+\bar{z}+\beta\delta\bar{J} - \min_{(\pi_i,\tilde{V}_i)}\left\{\sum_{i=1}^{2}\pi_i\upsilon^{-1}(\tilde{V}_i-\beta\bar{x}),\ \text{s.t.}\ \sum_{i=1}^{2}\pi_i\tilde{V}_i = V\right\}$$
$$\leqslant \bar{y}+\bar{z}+\beta\delta\bar{J} - \upsilon^{-1}(\underline{x}-\beta\bar{x}) \leqslant \bar{J},$$

where the first inequality uses the bounds on y, z, w and J, and the second inequality uses convexity of $\upsilon^{-1}(.)$. Also, the firm's value $\tilde{J}(V,y,z)$ is such that

$$\tilde{J}(V,y,z) \geqslant F(\gamma_0,V,y,z) \geqslant \underline{y}+\underline{z}-\upsilon^{-1}(\bar{x}-\beta\underline{x})+\beta\delta\underline{J} \geqslant \underline{J},$$

where the first inequality uses the fact that γ_0 belongs to Γ, and the second inequality uses the bounds on y, z, w and J.

(iii) In Appendix F, we prove that $\tilde{J}$ is concave with respect to V. Hence, $\tilde{J}$ satisfies property (J3) of the set $\mathcal{J}$. □

Now, consider two arbitrary functions J_n, $J_r \in \mathcal{J}$. Let θ_n, R_n, $\tilde{p}_n$, U_n, F_n and $\tilde{J}_n$ denote the functions computed with J_n. Let θ_r, R_r, $\tilde{p}_r$, U_r, F_r and $\tilde{J}_r$ denote the functions computed with J_r. The next lemma proves that the mapping T is continuous; that is, as the distance between J_n and J_r converges to zero, the distance between $\tilde{J}_n$ and $\tilde{J}_r$ converges to zero as well. The function $\tilde{J}$ depends on J through $(\theta, R, U, \tilde{p})$ and $(w, d, \hat{V})$. We have already established that θ, R, U and $\tilde{p}$ are all continuous in J. A main step in proving continuity of $\tilde{J}$ in J is to prove that the job-destruction probability d is continuous in J. The proof of the following lemma uses a constructive approach to establish this result.

Lemma 5.2. *For any $\rho > 0$ and any $J_n, J_r \in \mathcal{J}$, if $\|J_n - J_r\| < \rho$, then*

$$\|\tilde{J}_n - \tilde{J}_r\| < \beta\lambda_e\alpha_p(\rho)\bar{J} + \alpha_J\rho, \tag{5.4}$$

where

$$\alpha_J \equiv \alpha_w + \beta\big[(1+\lambda_e)(1+\bar{B}_J\alpha_{\hat{V}}) + \lambda_e\bar{B}_p\alpha_{\hat{V}}\bar{J}\big],$$
$$\alpha_{\hat{V}} \equiv (\lambda_e\alpha_R+\alpha_U+1)/(1-\lambda_e), \quad \alpha_w \equiv \beta(\alpha_U+\alpha_{\hat{V}}+\lambda_e\alpha_R)/\underline{\upsilon}'.$$

Proof. For the sake of brevity, suppress the dependence of various functions on $(\hat{y},\hat{z})$. Let $\rho > 0$ be an arbitrary real number. Let J_n and J_r be arbitrary functions in $\mathcal{J}$ such that $\|J_n - J_r\| < \rho$. Let (V,y,z) be an arbitrary point in $X \times Y \times Z$. Without loss in generality, assume that $\tilde{J}_n(V,y,z) \leqslant \tilde{J}_r(V,y,z)$. (If $\tilde{J}_n(V,y,z) > \tilde{J}_r(V,y,z)$, just switch the roles of $\tilde{J}_n$ and $\tilde{J}_r$ in the proof below.)

For $J = J_r$ in the firm's problem in (5.3), denote as $\gamma_r = (\pi_{1,r}, \tilde{V}_{1,r}, \hat{V}_{1,r}, \hat{V}_{2,r})$ a tuple such that $\gamma_r \in \Gamma$ is a solution to the firm's problem; i.e., $\tilde{J}_r(V,y,z) = F_r(\gamma_r,V,y,z)$. Let $w_{i,r}$ and $d_{i,r}$ be the wage and the separation probability implied by (5.3) with $\gamma = \gamma_r$ and $J = J_r$. For $J = J_n$ in the firm's problem in (5.3), consider a candidate choice $\gamma_n = (\pi_{1,n}, \tilde{V}_{1,n}, \hat{V}_{1,n}, \hat{V}_{2,n})$, where $\pi_{1,n} = \pi_{1,r}$, $\tilde{V}_{1,n} = \tilde{V}_{1,r}$, and

$$\hat{V}_{i,n} = \begin{cases} \hat{V}_{i,r}, & \text{if } [\hat{V}_{i,r}+\lambda_e R_n(\hat{V}_{i,r})-U_n][\hat{V}_{i,r}+\lambda_e R_r(\hat{V}_{i,r})-U_r] > 0, \\ U_n-\lambda_e R_n(\hat{V}_{i,n})+\rho, & \text{if } \hat{V}_{i,r}+\lambda_e R_n(\hat{V}_{i,r}) \leqslant U_n,\ \hat{V}_{i,r}+\lambda_e R_r(\hat{V}_{i,r}) \geqslant U_r, \\ U_n-\lambda_e R_n(\hat{V}_{i,n})-\rho, & \text{if } \hat{V}_{i,r}+\lambda_e R_n(\hat{V}_{i,r}) \geqslant U_n,\ \hat{V}_{i,r}+\lambda_e R_r(\hat{V}_{i,r}) < U_r. \end{cases} \tag{5.5}$$

Let $w_{i,n}$ and $d_{i,n}$ be the wage and separation probability implied by (5.3) with $\gamma = \gamma_n$ and $J = J_n$. Eq. (5.5) implies $d_{i,n} = d_{i,r}$. The choice γ_n, together with $(w_{i,n}, d_{i,n})$, is feasible for the firm when $J = J_n$, but may not necessarily be optimal. Thus, $F_n(\gamma_n, V, y, z) \leqslant \tilde{J}_n(V, y, z)$, and

$$(0 \leqslant) \quad \tilde{J}_r(V, y, z) - \tilde{J}_n(V, y, z) \leqslant F_r(\gamma_r, V, y, z) - F_n(\gamma_n, V, y, z).$$

We prove that the last difference is bounded by the RHS of (5.4).

First, we want to bound the distance $\|\hat{V}_{i,n} - \hat{V}_{i,r}\|$. To this aim, let $(\hat{y}, \hat{z})$ denote an arbitrary point in $Y \times Z$. Consider the case in which $\hat{V}_{i,r} + \lambda_e R_n(\hat{V}_{i,r}) - U_n$ has the same sign as $\hat{V}_{i,r} + \lambda_e R_r(\hat{V}_{i,r}) - U_r$. In this case, $\hat{V}_{i,n} = \hat{V}_{i,r}$ and, hence, $\|\hat{V}_{i,n} - \hat{V}_{i,r}\| < \alpha_{\hat{V}} \rho$. Next, consider the case in which $\hat{V}_{i,r} + \lambda_e R_n(\hat{V}_{i,r}) - U_n$ has a different sign from $\hat{V}_{i,r} + \lambda_e R_r(\hat{V}_{i,r}) - U_r$. In this case, the absolute value of $\hat{V}_{i,r} + \lambda_e R_n(\hat{V}_{i,r}) - U_n$ is such that

$$\begin{aligned} \left|\hat{V}_{i,r} + \lambda_e R_n(\hat{V}_{i,r}) - U_n\right| &\leqslant \left|\hat{V}_{i,r} + \lambda_e R_n(\hat{V}_{i,r}) - U_n - \left[\hat{V}_{i,r} + \lambda_e R_r(\hat{V}_{i,r}) - U_r\right]\right| \\ &\leqslant (\lambda_e \alpha_R + \alpha_U)\rho, \end{aligned} \tag{5.6}$$

where the second inequality uses the bounds in (4.9) and (4.12). Moreover, the absolute value of $\hat{V}_{i,r} + \lambda_e R_n(\hat{V}_{i,r}) - U_n$ is such that

$$\begin{aligned} \left|\hat{V}_{i,r} + \lambda_e R_n(\hat{V}_{i,r}) - U_n\right| &= \left|\hat{V}_{i,r} + \lambda_e R_n(\hat{V}_{i,r}) - U_n - \left[\hat{V}_{i,n} + \lambda_e R_n(\hat{V}_{i,n}) - U_n\right]\right| - \rho \\ &\geqslant (1 - \lambda_e)|\hat{V}_{i,r} - \hat{V}_{i,n}| - \rho, \end{aligned} \tag{5.7}$$

where the equality uses the definition of $\hat{V}_{i,n}$ in (5.5), and the inequality uses the bounds in (4.6). From (5.6) and (5.7), it follows that $(0 <)\ \hat{V}_{i,n} - \hat{V}_{i,r} < \alpha_{\hat{V}} \rho$ and, hence, $|\hat{V}_{i,n} - \hat{V}_{i,r}| < \alpha_{\hat{V}} \rho$. Since these results hold for all $(\hat{y}, \hat{z}) \in Y \times Z$, we have

$$\|\hat{V}_{i,n} - \hat{V}_{i,r}\| < \alpha_{\hat{V}} \rho. \tag{5.8}$$

Second, we want to bound the distance $|w_{i,r} - w_{i,n}|$. From the definitions of $w_{i,r}$ and $w_{i,n}$, it follows that $\upsilon(w_{i,r})$ and $\upsilon(w_{i,n})$ are

$$\begin{aligned} \upsilon(w_{i,r}) &= \tilde{V}_{i,r} - \beta \mathbb{E}_{\hat{s}}\left[d_{i,r} U_r + (1 - d_{i,r})\left(\hat{V}_{i,r} + \lambda_e R_r(\hat{V}_{i,r})\right)\right], \\ \upsilon(w_{i,n}) &= \tilde{V}_{i,n} - \beta \mathbb{E}_{\hat{s}}\left[d_{i,n} U_n + (1 - d_{i,n})\left(\hat{V}_{i,n} + \lambda_e R_n(\hat{V}_{i,n})\right)\right] \\ &= \tilde{V}_{i,r} - \beta \mathbb{E}_{\hat{s}}\left[d_{i,r} U_n + (1 - d_{i,r})\left(\hat{V}_{i,n} + \lambda_e R_n(\hat{V}_{i,n})\right)\right], \end{aligned}$$

where the last line uses the fact that, by construction, $\tilde{V}_{i,n} = \tilde{V}_{i,r}$ and $d_{i,n} = d_{i,r}$. From the previous equations, it follows that the distance between $\upsilon(w_{i,n})$ and $\upsilon(w_{i,r})$ is such that

$$\begin{aligned} \left|\upsilon(w_{i,n}) - \upsilon(w_{i,r})\right| &\geqslant \underline{\upsilon}' |w_{i,n} - w_{i,r}|, \\ \left|\upsilon(w_{i,n}) - \upsilon(w_{i,r})\right| &\leqslant \beta \mathbb{E}_{\hat{s}}\left\{|U_n - U_r| + \left|\left[\hat{V}_{i,n} + \lambda_e R_n(\hat{V}_{i,n})\right] - \left[\hat{V}_{i,r} + \lambda_e R_n(\hat{V}_{i,r})\right]\right|\right\} \\ &\quad + \beta \mathbb{E}_{\hat{s}}\left\{\left|\left[\hat{V}_{i,r} + \lambda_e R_n(\hat{V}_{i,r})\right] - \left[\hat{V}_{i,r} + \lambda_e R_r(\hat{V}_{i,r})\right]\right|\right\} \\ &< \beta(\alpha_U + \alpha_{\hat{V}} + \lambda_e \alpha_R)\rho, \end{aligned} \tag{5.9}$$

where the last inequality uses the bounds in (4.12), (5.8) and (4.9). Taken together, the two inequalities in (5.9) imply that

$$|w_{i,n} - w_{i,r}| < \alpha_w \rho. \tag{5.10}$$

Third, we want to bound the distance between $(1 - \lambda_e \tilde{p}_n(\hat{V}_{i,n})) J_n(\hat{V}_{i,n})$ and $(1 - \lambda_e \tilde{p}_r(\hat{V}_{i,r})) \times J_r(\hat{V}_{i,r})$. To this aim, note that the distance between $J_n(\hat{V}_{i,n})$ and $J_r(\hat{V}_{i,r})$ is such that

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$$\begin{aligned}\left|J_n(\hat{V}_{i,n}) - J_r(\hat{V}_{i,r})\right| &\leqslant \left|J_n(\hat{V}_{i,n}) - J_n(\hat{V}_{i,r})\right| + \left|J_n(\hat{V}_{i,r}) - J_r(\hat{V}_{i,r})\right| \\ &< (1 + \bar{B}_J \alpha_{\hat{V}})\rho, \end{aligned} \tag{5.11}$$

where the last inequality uses the bounds in (5.8). Also, note that the distance between $\tilde{p}_n(\hat{V}_{i,n})J_n(\hat{V}_{i,n})$ and $\tilde{p}_r(\hat{V}_{i,r})J_r(\hat{V}_{i,r})$ is such that

$$\begin{aligned}&\left|\tilde{p}_n(\hat{V}_{i,n})J_n(\hat{V}_{i,n}) - \tilde{p}_r(\hat{V}_{i,r})J_r(\hat{V}_{i,r})\right| \\ &\quad\leqslant \tilde{p}_n(\hat{V}_{i,n})\left|J_n(\hat{V}_{i,n}) - J_r(\hat{V}_{i,n})\right| + \tilde{p}_n(\hat{V}_{i,n})\left|J_r(\hat{V}_{i,n}) - J_r(\hat{V}_{i,r})\right| \\ &\qquad + \left|J_r(\hat{V}_{i,r})\right|\left|\tilde{p}_n(\hat{V}_{i,n}) - \tilde{p}_n(\hat{V}_{i,r})\right| + \left|J_r(\hat{V}_{i,r})\right|\left|\tilde{p}_n(\hat{V}_{i,r}) - \tilde{p}_r(\hat{V}_{i,r})\right| \\ &\quad< (1 + \bar{B}_J\alpha_{\hat{V}} + \bar{B}_p\alpha_{\hat{V}}\bar{J})\rho + \alpha_p(\rho)\bar{J}, \end{aligned} \tag{5.12}$$

where we have used Lemma 4.10 to bound the last difference. (5.11) and (5.12) imply:

$$\begin{aligned}&\left|\big(1 - \lambda_e\tilde{p}_n(\hat{V}_{i,n})\big)J_n(\hat{V}_{i,n}) - \big(1 - \lambda_e\tilde{p}_r(\hat{V}_{i,r})\big)J_r(\hat{V}_{i,r})\right| \\ &\quad\leqslant \left|J_n(\hat{V}_{i,n}) - J_r(\hat{V}_{i,r})\right| + \lambda_e\left|\tilde{p}_n(\hat{V}_{i,n})J_n(\hat{V}_{i,n}) - \tilde{p}_r(\hat{V}_{i,r})J_r(\hat{V}_{i,r})\right| \\ &\quad< \lambda_e\alpha_p(\rho)\bar{J} + \big[(1+\lambda_e)(1 + \bar{B}_J\alpha_{\hat{V}}) + \lambda_e\bar{B}_p\alpha_{\hat{V}}\bar{J}\big]\rho. \end{aligned} \tag{5.13}$$

Finally, we prove that the difference, $F_r(\gamma_r, V, y, z) - F_n(\gamma_n, V, y, z)$, is bounded by the RHS of (5.4). From the bounds (5.8), (5.10) and (5.13), it follows that

$$\begin{aligned}0 &\leqslant \tilde{J}_r(V, y, z) - \tilde{J}_n(V, y, z) \leqslant F_r(\gamma_r, V, y, z) - F_n(\gamma_n, V, y, z) \\ &\leqslant \sum_{i=1}^{2}\pi_{i,r}\Big\{|w_{i,n} - w_{i,r}| + \beta\mathbb{E}_{\hat{s}}\Big[\Big|\big(1 - \lambda_e\tilde{p}_n(\hat{V}_{i,n})\big)J_n(\hat{V}_{i,n}) - \big(1 - \lambda_e\tilde{p}_r(\hat{V}_{i,r})\big)J_r(\hat{V}_{i,r})\Big|\Big]\Big\} \\ &< \beta\lambda_e\alpha_p(\rho)\bar{J} + \big\{\alpha_w + \beta\big[(1+\lambda_e)(1 + \bar{B}_J\alpha_{\hat{V}}) + \lambda_e\bar{B}_p\alpha_{\hat{V}}\bar{J}\big]\big\}\rho = \beta\lambda_e\alpha_p(\rho)\bar{J} + \alpha_J\rho.\end{aligned}$$

Since the above inequality holds for all $(V, y, z) \in X \times Y \times Z$, it implies the result stated in the lemma. □

5.2. *Existence of a BRE with dynamic contracts*

Now, we are in the position to establish the paper's main result.

Theorem 5.3. *There exists a BRE with dynamic contracts.*

Proof. First, fix $\varepsilon > 0$ to be an arbitrary real number. Let ρ_ε be the unique positive solution for ρ of the equation $\beta\lambda_e\alpha_p(\rho)\bar{J} + \alpha_j\rho = \varepsilon$. For all $J_n, J_r \in \mathcal{J}$ such that $\|J_n - J_r\| < \rho_\varepsilon$, Lemma 5.2 implies that $\|TJ_n - TJ_r\| < \varepsilon$. Hence, the equilibrium operator T is continuous. Next, let ρ_y denote the minimum distance between distinct elements of the set Y, and let ρ_z be the minimum distance between distinct elements of the set Z, i.e. $\rho_y = \min_Y |y_i - y_j|$ and $\rho_z = \min_Z |z_i - z_j|$.[20] Also, let $\|.\|_E$ denote the standard norm on the Euclidean space $X \times Y \times Z$. Let $\tilde{\rho}_\varepsilon = \min\{\underline{\upsilon}'\varepsilon, \rho_y, \rho_z\}$. For all $(V_1, y_1, z_1), (V_2, y_2, z_2) \in X \times Y \times Z$ such that $\|(V_2, y_2, z_2) - (V_1, y_1, z_1)\|_E < \tilde{\rho}_\varepsilon$ and all $J \in \mathcal{J}$, Lemma 5.1 implies that TJ satisfies the property (J1) of the set $\mathcal{J}$ and, consequently, $|(TJ)(V_2, y_2, z_2) - (TJ)(V_2, y_2, z_2)| < \varepsilon$. Hence, the family of

[20] If Y contains only one element, we can set $\rho_y = 1$. Similarly, if Z contains only one element, set $\rho_z = 1$.

functions $T(\mathcal{J})$ is equicontinuous. Finally, Lemma 5.1 implies that the equilibrium operator T maps the set of functions $\mathcal{J}$ into itself.

From these properties, it follows that the equilibrium operator T satisfies the conditions of Schauder's fixed point theorem [28, Theorem 17.4]. Therefore, there exists a firm's value function $J^* \in \mathcal{J}$ such that $TJ^* = J^*$. Denote as θ^* the market tightness function computed with J^*. Denote as R^* and m^* the search value and policy functions computed with θ^*. Denote as U^* the unemployment value function computed with R^*. Denote as c^* the contract policy function computed with J^*, θ^*, R^*, m^*, and U^*. The functions $\{\theta^*, R^*, m^*, U^*, J^*, c^*\}$ satisfy the conditions (i)–(v) in the definition of a recursive equilibrium. The functions $\{\theta^*, R^*, m^*, U^*, J^*, c^*\}$ depend on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y, and not through the distribution of workers across different employment states, (u, g). Hence, the functions $\{\theta^*, R^*, m^*, U^*, J^*, c^*\}$ constitute a BRE. □

Directed search is necessary for existence of a BRE. To see this necessity clearly, suppose that search is random, instead. Then the equilibrium condition (2.7) is replaced by

$$k \geqslant \max_{x \in X} q\big(\theta(\psi)\big)\mathbb{I}(x, \psi) J(x, \psi, z_0), \tag{5.14}$$

and $\theta(\psi) \geqslant 0$, with complementary slackness. The term on the LHS of (5.14) is the cost of creating a vacancy. The expression on the RHS of (5.14) is the maximized benefit of creating a vacancy. The first term on the RHS is the probability that a firm meets a worker. The second term denotes the probability that a worker met by a firm is willing to accept an employment contract that provides him with the lifetime utility x. The third term is the value to the firm of being matched with a worker to whom it has promised the lifetime utility x. With random search, the worker who meets the firm is a random draw from the distribution of workers over the values, and so a worker's acceptance probability of a new match depends on the distribution of workers across employment states. That is, the dependence of $\mathbb{I}(x, \psi)$ on g is not trivial. In this case, the equilibrium condition (5.14) holds only if the distribution affects also the equilibrium market tightness or the firm's value function. In either case, the equilibrium fails to be block recursive with random search. In contrast, directed search eliminates the dependence of the acceptance probability on the distribution of workers because a worker always accepts a job that he chooses to search for; that is, $\mathbb{I}(x^*, \psi) = 1$ where $x^* = m(V, \psi)$.[21]

For the sake of completeness, let us list three other assumptions about the production technology and the search process that are necessary for existence of a BRE: the linear production function, the vacancy cost independent of the aggregate vacancy rate, and a matching technology with constant returns to scale. If the production function were either concave or convex, the distribution of workers across different employment states would affect the output of a match and, in turn, the firm's value function, the market tightness function and the value of unemployment. If the vacancy cost depends the aggregate vacancy rate, the distribution of workers across different employment states would affect the aggregate vacancy rate, the vacancy cost and, ultimately, the equilibrium market tightness. Finally, if the matching process between vacancies and applicants

[21] Eq. (5.14) implies that the equilibrium cannot be block recursive with random search on the job, but it does not imply that such an equilibrium is always difficult to compute. For example, if the model happens to be such that the distribution of workers across employment states varies in a simple way in response to aggregate productivity shocks, then computing the equilibrium is possible even without block recursivity. Indeed, this is the property of the distribution that allows Moscarini and Postel-Vinay [20] to compute the stochastic equilibrium of a model of random search on the job.

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exhibits non-constant returns to scale, the distribution of applicants across different submarkets (and, hence, the distribution of workers across different employment states) would affect the market tightness function and, in turn, the firm's and worker's value functions. We emphasize that these assumptions are standard. For example, they are maintained in the models of search on the job by Burdett and Mortensen [5], Postel-Vinay and Robin [23], and Burdett and Coles [4], where the equilibrium fails to be block recursive because search is undirected.[22]

6. BRE with fixed-wage contracts

In the model with fixed-wage contracts, the equilibrium operator T may not be continuous. For example, the search value function, R_n, and the unemployment value function, U_n, computed with the firm's value function J_n may be such that the worker prefers being employed at the wage w than being unemployed. However, given a different value function J_r that is arbitrarily close to J_n, the search value function, R_r, and the unemployment value function, U_r, may be such that the worker prefers unemployment to employment. In this case, the probability that a worker leaves a job that pays the wage w is not continuous in J and, hence, the firm's value from employing a worker at the wage w, $K(w, s)$ defined in (2.5), and the firm's updated value function, TJ, are not continuous in J.[23]

Since the equilibrium operator T may not be continuous, we cannot directly appeal to Schauder's theorem in order to prove existence of a fixed point of T and, in turn, existence of a BRE. Instead, we adopt the following strategy. We consider a proxy of the model with fixed-wage contracts in which a worker is not allowed to voluntarily quit his jobs during the separation stage. Formally, in this proxy model, the equilibrium conditions (2.4) and (2.5) are replaced by

$$H(w,\psi) = w + \beta \mathbb{E}_{\hat{\psi}} \big\{ \delta U(\hat{\psi}) + (1-\delta)\big[H(w,\hat{\psi}) + \lambda_e \max\big\{0, R\big(H(w,\hat{\psi}),\hat{\psi}\big)\big\}\big]\big\}, \tag{6.1}$$

and

$$K(w,s) = y + z - w + \beta(1-\delta)\mathbb{E}_{\hat{s}}\big[\big(1 - \lambda_e \tilde{p}\big(H(w,\hat{\psi}),\hat{\psi}\big)\big)K(w,\hat{s})\big]. \tag{6.2}$$

We prove that the equilibrium operator associated with the proxy model admits a fixed point because it satisfies all the conditions of Schauder's theorem (including continuity). We use the fixed point to construct a BRE of the proxy model. If, along the equilibrium path, a worker never has the incentive to quit his job during the separation stage, the BRE of the proxy model is also a BRE of the original model.

[22] Since search in reality may be a mix of directed and random search, the reader may wonder whether the BRE of our model is robust to the introduction of an ε-amount of random search. More specifically, the reader may wonder whether the BRE of the model with directed search is the limit of a sequence of equilibria in which the worker's job application is directed with probability $1-\varepsilon$, and random with probability ε, $\varepsilon \to 0$. If the equilibrium mapping T is a contraction, it is not difficult to prove that the BRE of the directed search model is the limit of a sequence of equilibria of the perturbed model. If the equilibrium mapping T is not a contraction, establishing the robustness of the BRE is more difficult and will be left for future research.

[23] This discontinuity does not occur with dynamic contracts, because the future wage path (i.e., the promised future value) can be adjusted to ensure that job separation rates are close to each other whenever the firm's value functions are close to each other. See the proof of Lemma 5.2.

6.1. Employment value

Given an arbitrary value function of a firm, $J \in \mathcal{J}$, let R denote the search value function that solves the equilibrium condition (2.1), and U the unemployment value function that solves the equilibrium condition (2.2). Given R and U, an employment value function is a solution to the equilibrium condition (6.1) if and only if it is a fixed point of the mapping T_H defined as

$$(T_H\varphi)(w,\psi) = w + \beta\mathbb{E}_{\hat{\psi}}\big\{\delta U(\hat{y}) + (1-\delta)\big[\varphi(w,\hat{\psi}) + \lambda_e \max\big\{0, R\big(\varphi(w,\hat{\psi}),\hat{y}\big)\big\}\big]\big\}. \tag{6.3}$$

In Lemma 6.1, we prove that there exists a unique fixed point of the mapping T_H within the set $\mathcal{C}(W \times Y)$ of bounded continuous functions $\varphi : W \times Y \to \mathbb{R}$ (where W is defined below). Therefore, there exists a unique employment value function, H, that satisfies the equilibrium condition (6.1), and depends on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y. Moreover, in Lemma 6.1, we prove that H is strictly decreasing and bi-Lipschitz continuous in w.

Lemma 6.1. *Let $W = [\underline{w}, \bar{w}]$, where $\bar{w}$ is given by $[1-\beta(1-\delta)]\bar{x} - \beta\delta\underline{U}$ and $\underline{w}$ by $\underline{x} - \beta[1-\beta(1-\delta)]^{-1}(\bar{w} + \beta\delta\bar{U})$.*

(i) *There exists a unique function $H \in \mathcal{C}(W \times Y)$ such that $H = T_H H$.*
(ii) *For all $y \in Y$ and all $w_1, w_2 \in W$, $w_1 \leqslant w_2$, H is such that*

$$w_2 - w_1 \leqslant H(w_2, y) - H(w_1, y) \leqslant (w_2 - w_1)/\big[1-\beta(1-\delta)\big]. \tag{6.4}$$

(iii) *For all $y \in Y$, H is such that*

$$H(\underline{w}, y) \leqslant \underline{x}, \qquad \bar{x} \leqslant H(\bar{w}, y), \quad \text{all } y \in Y. \tag{6.5}$$

Proof. In Appendix D. □

From the properties of the employment value function, H, we can derive some properties of the wage function, h, which is the solution of the equation $H(w,\psi) = V$ with respect to w. First, since H is strictly increasing in w, h is well-defined. Second, since H is strictly increasing and bi-Lipschitz continuous in w, h is strictly increasing and bi-Lipschitz in V. More specifically, for all $y \in Y$ and all $V_1, V_2 \in X$, with $V_1 \leqslant V_2$, we have

$$\big[1-\beta(1-\delta)\big](V_2 - V_1) \leqslant h(V_2, y) - h(V_1, y) \leqslant V_2 - V_1. \tag{6.6}$$

Finally, since H is strictly increasing in w and satisfies property (6.5), $h(V, y)$ belongs to the interval W for all $(V, y) \in X \times Y$.

Now, consider two arbitrary functions $J_n, J_r \in \mathcal{J}$. Let R_n, U_n, H_n and h_n denote the functions computed with J_n. Similarly, let R_r, U_r, H_r and h_r denote the functions computed with $J_r \in \mathcal{J}$. Lemma 6.2 proves that as the distance between J_n and J_r converges to zero, the distance between H_n and H_r and the distance between h_n and h_r both converge to zero. That is, H and h are continuous in J.

Lemma 6.2. *For any $\rho > 0$ and any $J_n, J_r \in \mathcal{J}$, if $\|J_n - J_r\| < \rho$, then*

$$\|H_n - H_r\| < \alpha_h\rho, \qquad \|h_n - h_r\| < \alpha_h\rho,$$
$$\alpha_h \equiv \beta(\alpha_u + \lambda_e\alpha_R)/(1-\beta). \tag{6.7}$$

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Proof. Let $\rho > 0$ be an arbitrary real number; let J_n and J_r be arbitrary functions in $\mathcal{J}$ such that $\|J_n - J_r\| < \rho$. Let (w, y) be an arbitrary point in $W \times Y$. Then, the distance between $H_n(w, y)$ and $H_r(w, y)$ is such that

$$\begin{aligned}
&\big|H_n(w, y) - H_r(w, y)\big| \\
&\quad \leqslant \beta \mathbb{E}_{\hat{y}}\big\{\big|U_n(y) - U_r(y)\big| + \lambda_e \big|\max\big\{0, R_n\big(H_n(w, \hat{y}), \hat{y}\big)\big\} - \max\big\{0, R_r\big(H_n(w, \hat{y}), \hat{y}\big)\big\}\big|\big\} \\
&\qquad + \beta \mathbb{E}_{\hat{y}}\big\{\big|H_n(w, \hat{y}) + \lambda_e \max\big\{0, R_r\big(H_n(w, \hat{y}), \hat{y}\big)\big\} - H_r(w, \hat{y}) \\
&\qquad - \lambda_e \max\big\{0, R_r\big(H_r(w, \hat{y}), \hat{y}\big)\big\}\big|\big\} \\
&\quad < \beta(\alpha_u + \lambda_e \alpha_R)\rho + \beta\|H_n - H_r\|,
\end{aligned}$$

where the last inequality uses the bounds in (4.12), (4.9), and (4.6). Since the above result holds for all $(w, y) \in W \times Y$, the RHS is an upper bound on $\|H_n - H_r\|$. Re-arranging terms yields the bound on $\|H_n - H_r\|$ given by (6.7).

Now, let (V, y) be an arbitrary point in $X \times Y$. The distance between $h_n(V, y)$ and $h_r(V, y)$ is such that

$$\begin{aligned}
\big|h_n(V, y) - h_r(V, y)\big| &\leqslant \big|H_n\big(h_n(V, y), y\big) - H_n\big(h_r(V, y), y\big)\big| \\
&= \big|H_r\big(h_r(V, y), y\big) - H_n\big(h_r(V, y), y\big)\big| < \alpha_h \rho,
\end{aligned}$$

where the first inequality uses the fact that $H_n(w, y)$ satisfies condition (6.4), and the equality uses the fact that $H_n(h_n(V, y), y) = H_r(h_r(V, y), y) = V$. Since the above result holds for all $(V, y) \in X \times Y$, the RHS is an upper bound on $\|h_n - h_r\|$, as given by (6.7). □

6.2. *Value function of the firm*

Let H and $\tilde{p}$ denote the employment value function and the separation probability computed with an arbitrary function $J \in \mathcal{J}$. Given H and $\tilde{p}$, a firm's value function is a solution to the equilibrium condition (6.2) if and only if it is a fixed point of the mapping T_K defined as

$$(T_K \varphi)(w, s) = y + z - w + \beta(1 - \delta)\mathbb{E}_{\hat{s}}\big[\big(1 - \lambda_e \tilde{p}\big(H(w, \hat{y}), \hat{y}\big)\big)\varphi(w, \hat{s})\big]. \tag{6.8}$$

In Lemma 6.3, we prove that there exists a unique fixed point of the mapping T_K within the set $\mathcal{C}(W \times Y \times Z)$ of bounded continuous functions $\varphi : W \times Y \times Z \to \mathbb{R}$. Therefore, there exists a unique value function of the firm, K, that satisfies the equilibrium condition (6.2), and that depends on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y. Then, we prove that K is bounded between $\underline{K}$ and $\bar{K}$, where

$$-\underline{K} = \bar{K} = \max\left\{\frac{|\underline{y} + \underline{z} - \bar{w}|}{1 - \beta(1 - \delta)}, \frac{|\bar{y} + \bar{z} - \underline{w}|}{1 - \beta(1 - \delta)}\right\}.$$

Finally, we prove that K is bi-Lipschitz continuous in w. That is, for all $w_1 \leqslant w_2$, the difference $K(w_2, y, z) - K(w_1, y, z)$ is bounded between $-\bar{B}_K(w_2 - w_1)$ and $-\underline{B}_K(w_2 - w_1)$, where

$$\underline{B}_K = \frac{1 - \beta(1 - \delta)(1 + \lambda_e \bar{B}_p \bar{K})}{[1 - \beta(1 - \delta)][1 - \beta(1 - \delta)(1 - \lambda_e)]}, \qquad \bar{B}_K = \frac{1 - \beta(1 - \delta)(1 + \lambda_e \bar{B}_p \underline{K})}{[1 - \beta(1 - \delta)]^2}.$$

In the remainder of this section, we will assume that the parameters of the model are such that $0 < \underline{B}_K \leqslant \bar{B}_K < \infty$.[24]

[24] One can verify that the condition $0 < \underline{B}_K \leqslant \bar{B}_K < \infty$ is satisfied as long as the probability λ_e that an employed worker has the opportunity of searching is not too large.

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Lemma 6.3.

(i) *There exists a unique function* $K \in \mathcal{C}(W \times Y \times Z)$ *such that* $K = T_K K$.
(ii) *For all* $(y, z) \in Y \times Z$ *and all* $w_1, w_2 \in W$, *with* $w_1 \leqslant w_2$, K *is such that*

$$-\bar{B}_K(w_2 - w_1) \leqslant K(w_2, y, z) - K(w_1, y, z) \leqslant -\underline{B}_K(w_2 - w_1). \tag{6.9}$$

(iii) *For all* $(w, y, z) \in W \times Y \times Z$, K *is such that*

$$K(w, y, z) \in [\underline{K}, \bar{K}]. \tag{6.10}$$

Proof. In Appendix E. □

Now, consider two arbitrary functions J_n, $J_r \in \mathcal{J}$. Let R_n, U_n, H_n, h_n and K_n denote the functions computed with J_n. Similarly, let R_r, U_r, H_r, h_r and K_r denote the functions computed with $J_r \in \mathcal{J}$. Lemma 6.4 proves that as the distance between J_n and J_r converges to zero, the distance between K_n and K_r goes to zero as well.

Lemma 6.4. *For any* $\rho > 0$ *and any* $J_n, J_r \in \mathcal{J}$, *if* $\|J_n - J_r\| < \rho$, *then*

$$\begin{aligned}
&\|K_n - K_r\| < \alpha_K(\rho), \\
&\alpha_K(\rho) \equiv \beta(1-\delta)\lambda_e \bar{K}\big(\bar{B}_p \alpha_h \rho + \alpha_p(\rho)\big)\big/\big[1 - \beta(1-\delta)\big].
\end{aligned} \tag{6.11}$$

Proof. Let $\rho > 0$ be an arbitrary real number; let J_n and J_r be arbitrary functions in $\mathcal{J}$ such that $\|J_n - J_r\| < \rho$. Let (w, y, z) be an arbitrary point in $W \times Y \times Z$. The distance between $K_n(w, y, z)$ and $K_r(w, y, z)$ is such that

$$\begin{aligned}
&\big|K_n(w, y, z) - K_r(w, y, z)\big| \\
&\quad \leqslant \beta(1-\delta)\mathbb{E}_{\hat{s}}\big\{\big|K_n(w, y, z) - K_r(w, y, z)\big|\big\} \\
&\qquad + \beta(1-\delta)\lambda_e \bar{K}\mathbb{E}_{\hat{s}}\big\{\big|\tilde{p}_n\big(H_n(w, \hat{y}), \hat{y}\big) - \tilde{p}_n\big(H_r(w, \hat{y}), \hat{y}\big)\big| \\
&\qquad + \big|\tilde{p}_n\big(H_r(w, \hat{y}), \hat{y}\big) - \tilde{p}_r\big(H_r(w, \hat{y}), \hat{y}\big)\big|\big\} \\
&\quad < \beta(1-\delta)\big[\|K_n - K_r\| + \lambda_e \bar{K}\big(B_p \alpha_h \rho + \alpha_p(\rho)\big)\big],
\end{aligned}$$

where the last inequality uses the bounds in (6.7), (6.9) and (4.4). Since this result holds for all $(w, y, z) \in W \times Y \times Z$, the RHS is an upper bound on $\|K_n - K_r\|$. Re-arranging terms yields the bound on $\|K_n - K_r\|$ given by (6.11). □

6.3. Existence of a BRE with fixed-wage contracts

In the previous subsections, we have computed the employment value function, H, the wage function, h, and the firm's value function, K, associated with an arbitrary $J \in \mathcal{J}$. In this subsection, we insert K and h into the right-hand side of the equilibrium condition (2.6), and we compute an update, $\tilde{J} = TJ$, for the value function J. More specifically, $\tilde{J}$ is given by

$$\tilde{J}(V, y, z) = \max_{\pi_i, \tilde{V}_i} \sum_{i=1}^{2} \pi_i K\big(h(\tilde{V}_i, y), y, z\big),$$

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$$\text{s.t.} \quad V = \sum_{i=1}^{2} \pi_i \tilde{V}_i,$$
$$\pi_i \in [0,1], \quad \pi_1 + \pi_2 = 1, \quad \tilde{V}_i \in X. \tag{6.12}$$

The updated function, $\tilde{J}$, has four properties. First, $\tilde{J}$ depends on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y. This property follows immediately from the fact that both the objective function and the choice set on the right-hand side of (5.1) depend on ψ only through y. Second, the updated value function, $\tilde{J}$, is bi-Lipschitz continuous in V. More specifically, for all $(y,z) \in Y \times Z$ and all $V_1, V_2 \in X$, with $V_1 \leqslant V_2$, the difference $\tilde{J}(V_2,y,z) - \tilde{J}(V_1,y,z)$ is bounded between $-\bar{B}_K(V_2 - V_1)$ and $-\underline{B}_K(1-\beta(1-\delta))(V_2 - V_1)$ (see part (i) in the proof of Lemma 6.5). Third, $\tilde{J}$ is bounded in $[\underline{J}, \bar{J}]$ for some bounds $\underline{J}$ and $\bar{J}$ that are independent of J and $\tilde{J}$. More specifically, for all $(V,y,z) \in X \times Y \times Z$, $\tilde{J}(V,y,z)$ is greater than $\underline{K}$ and smaller than $\bar{K}$ (see part (ii) in the proof of Lemma 6.5). Finally, $\tilde{J}$ is concave in V (see part (iii) in the proof of Lemma 6.5). Therefore, given the appropriate choices of $\underline{B}_J$, $\bar{B}_J$, $\underline{J}$, and $\bar{J}$, the updated value function, $\tilde{J}$, satisfies conditions (J1), (J2) and (J3) and, hence, it belongs to the set $\mathcal{J}$. This argument is formalized in the following lemma:

Lemma 6.5. *Set $\underline{J} = \underline{K}$ and $\bar{J} = \bar{K}$. Set $\underline{B}_J = \underline{B}_K(1-\beta(1-\delta))$ and $\bar{B}_J = \bar{B}_K$. Then, the updated value function, $\tilde{J}$, belongs to the set $\mathcal{J}$.*

Proof. (i) Let (V,y,z) be an arbitrary point in $X \times Y \times Z$. Then, $\tilde{J}(V,y,z)$ is such that

$$\tilde{J}(V,y,z) \leqslant \max_{\tilde{V}_1 \in X} K\big(h(\tilde{V}_1,y),y,z\big) \leqslant \max_{w \in W} K(w,y,z) \leqslant \bar{K},$$
$$\tilde{J}(V,y,z) \geqslant \min_{\tilde{V}_1 \in X} K\big(h(\tilde{V}_1,y),y,z\big) \geqslant \min_{w \in W} K(w,y,z) \geqslant \underline{K},$$

where we used the fact that if $\tilde{V}_1 \in X$ then $h(\tilde{V}_1,y) \in W$. The above inequalities imply that $\tilde{J}$ satisfies property (J1) of the set $\mathcal{J}$.

(ii) Let (y,z) be an arbitrary point in $Y \times Z$, and V_1, V_2 two arbitrary points in X, with $V_1 \leqslant V_2$. Let $\{\pi_{i,1}, \tilde{V}_{i,1}\}_{i=1}^2$ denote the maximizer of (6.12) for $V = V_1$, and $\{\pi_{i,2}, \tilde{V}_{i,2}\}_{i=1}^2$ the maximizer of (6.12) for $V = V_2$. Let $\{\Delta_{i,1}\}_{i=1}^2$ be a vector such that $\Sigma_{i=1}^2 \pi_{i,1}(\tilde{V}_{i,1} + \Delta_{i,1}) = V_2$ and $\Delta_{i,1} \in [0, \bar{x} - V_{i,1}]$. Also, let $\{\Delta_{i,2}\}_{i=1}^2$ be a vector such that $\Sigma_{i=1}^2 \pi_{i,2}(\tilde{V}_{i,2} - \Delta_{i,2}) = V_1$ and $\Delta_{i,2} \in [0, V_{i,2} - \underline{x}]$. Note that $\{\pi_{i,1}, \tilde{V}_{i,1} + \Delta_{i,1}\}_{i=1}^2$ belongs to the choice set of (6.12) for $V = V_2$. Therefore,

$$\begin{aligned}
\tilde{J}(V_2,y,z) - \tilde{J}(V_1,y,z) &\geqslant \sum_{i=1}^{2} \pi_{i,1}\big[K\big(h(\tilde{V}_{i,1} + \Delta_{i,1},y),y,z\big) - K\big(h(\tilde{V}_{i,1},y),y,z\big)\big] \\
&\geqslant -\bar{B}_K\left[\sum_{i=1}^{2} \pi_{i,1}\big(h(\tilde{V}_{i,1} + \Delta_{i,1},y) - h(\tilde{V}_{i,1},y)\big)\right] \\
&= -\bar{B}_K(V_2 - V_1).
\end{aligned}$$

Next, note that $\{\pi_{i,2}, \tilde{V}_{i,2} - \Delta_{i,2}\}_{i=1}^2$ belongs to the choice set of (6.12) for $V = V_2$. Therefore,

$$\tilde{J}(V_2,y,z) - \tilde{J}(V_1,y,z) \leqslant \sum_{i=1}^{2} \pi_{i,2}\big[K\big(h(\tilde{V}_{i,2},y),y,z\big) - K\big(h(\tilde{V}_{i,2} - \Delta_{i,2},y),y,z\big)\big]$$

$$\leqslant -\underline{B}_K\left[\sum_{i=1}^{2}\pi_{i,2}\big(h(\tilde{V}_{i,2},y)-h(\tilde{V}_{i,2}-\Delta_{i,2},y)\big)\right]$$
$$=-\underline{B}_K\big(1-\beta(1-\delta)\big)(V_2-V_1).$$

The above inequalities imply that $\tilde{J}$ satisfies property (J2) of the set $\mathcal{J}$.

(iii) Finally, Appendix F shows that $\tilde{J}$ is concave with respect to V. Hence, $\tilde{J}$ satisfies property (J3) of the set $\mathcal{J}$. □

Now, consider two arbitrary functions $J_n, J_r \in \mathcal{J}$. Let H_n, h_n, K_n and $\tilde{J}_n$ denote the functions computed with J_n. Similarly, let H_r, h_r, K_r and $\tilde{J}_r$ denote the functions computed with $J_r \in \mathcal{J}$. Lemma 6.4 proves that as the distance between J_n and J_r converges to zero, the distance between $\tilde{J}_n$ and $\tilde{J}_r$ goes to zero as well.

Lemma 6.6. *For any $\rho > 0$ and any $J_n, J_r \in \mathcal{J}$, if $\|J_n - J_r\| < \rho$, then*

$$\|\tilde{J}_n-\tilde{J}_r\|<\alpha_J(\rho),\quad \alpha_J(\rho)\equiv\alpha_K(\rho)+\bar{B}_K\alpha_h\rho. \tag{6.13}$$

Proof. Let $\rho > 0$ be an arbitrary real number; let J_n and J_r be arbitrary functions in $\mathcal{J}$ such that $\|J_n - J_r\| < \rho$. Denote as H_n, h_n and K_n the functions computed with J_n, and H_r h_r and K_r the functions computed with J_r. Let (V, y, z) be an arbitrary point in $X \times Y \times Z$. The distance between $\tilde{J}_n(V, y, z)$ and $\tilde{J}_r(V, y, z)$ is such that

$$\big|\tilde{J}_n(V,y,z)-\tilde{J}_r(V,y,z)\big|$$
$$\leqslant \max_{\tilde{V}_1\in X}\big[\big|K_n\big(h_n(\tilde{V}_1)\big)-K_n\big(h_r(\tilde{V}_1)\big)\big|+\big|K_n\big(h_r(\tilde{V}_1)\big)-K_r\big(h_r(\tilde{V}_1)\big)\big|\big]$$
$$\leqslant \max_{V_1\in X}\big[\bar{B}_K\|h_n-h_r\|+\|K_n-K_r\|\big]\leqslant \alpha_K(\rho)+\bar{B}_K\alpha_h\rho,$$

where the last inequality uses the bounds in (6.7), (6.9) and (6.11). Since this result holds for all $(V, y, z) \in X \times Y \times Z$, the RHS is an upper bound on $\|\tilde{J}_n - \tilde{J}_r\|$. □

Lemma 6.5 implies that the equilibrium operator T maps the set $\mathcal{J}$ into itself. Moreover, since the functions in the set $\mathcal{J}$ are bi-Lipschitz and the sets Y and Z are finite, Lemma 6.5 implies that the family of functions $T(\mathcal{J})$ is equicontinuous. In addition, Lemma 6.6 implies that the operator T is continuous. Since these properties of the operator T are sufficient to apply Schauder's fixed point theorem, there exists a function $J^* \in \mathcal{J}$ such that $TJ^* = J^*$. The firm's value function J^*, the associated tightness function θ^*, search value function R^*, search policy function m^*, and unemployment value function U^*, constitute a BRE. This completes the proof of the following theorem:

Theorem 6.7. *There exists a BRE for the proxy of the model with fixed-wage contracts.*

For any BRE of the proxy model, we can compute the worker's value of unemployment, $U^*(y)$, and the worker's value of employment at the beginning of the search stage, $H^*(w, y) + \lambda_e \max\{0, R^*(H^*(w, y), y)\}$. A BRE of the proxy model is a BRE of the original model if

$$U^*(y)\leqslant H^*(w,y)+\lambda_e\max\big\{0,R^*\big(H^*(w,y),y\big)\big\} \tag{6.14}$$

for all equilibrium wages w and for all realizations of the aggregate component of productivity y. This condition implicitly restricts the parameter values of the model. We do not explicitly characterize this restriction here. However, notice that, since unemployed workers search for jobs that offer lifetime utility $H^*(w, y)$ greater than $U^*(y)$ and since employed workers search for even better jobs, (6.14) is likely to be satisfied as long as the dispersion in the realizations of aggregate productivity shocks is sufficiently small. This is the case in the calibrated example below.

7. A calibrated example

In Sections 5 and 6, we have established existence of a BRE in a stochastic model of directed search on the job. In this section, we illustrate additional properties of a BRE by calibrating the model to the data on the US labor market. Given the calibrated parameters, we construct the equilibrium operator T and we apply it to an arbitrary value function, $J \in \mathcal{J}$, until we reach a fixed point, J^*. Then, we construct a BRE by computing the agents' value functions, policy functions and the market tightness function associated with J^*. For the sake of brevity, we report our findings only for the version of the model with fixed-wage contracts.

The parameters in workers' preferences are the discount factor, β, and the value of leisure, b. The parameters in the search technology are the probability that an unemployed worker is able to search, λ_u, the probability that an employed worker is able to search, λ_e, and the parameters in the job-finding probability function, $p(\theta)$. We assume that $p(\theta) = \theta(1+\theta^\gamma)^{-1/\gamma}$. The parameters in the production technology are the vacancy cost, k, the exogenous job-destruction probability, δ, and the parameters in the stochastic processes for the idiosyncratic and the aggregate components of productivity. We assume that the idiosyncratic component of productivity, z, is always equal to zero, and that the aggregate component of productivity, y, obeys a two-state Markov process, with $y \in \{0.95, 1.05\}$. The unconditional mean of y is normalized to 1.

We set the model period to be one quarter. We set β equal to 0.987, so that the annual interest rate in the model is 5 percent. We set k, δ, and λ_e equal to 10^{-7}, 0.045, and 0.3 respectively, so that the average transition rates between employment, unemployment, and across employers are the same in the model as in the US data.[25] We normalize λ_u to 1. We tentatively set γ to 0.2, which implies an elasticity of substitution between vacancies and applicants of 5/6. Finally, we set b equal to 0.7, so that the consumption value of leisure is 70 percent of the consumption value of work (a figure that is empirically supported by [10]).

Given these parameter values, we compute a BRE of the proxy model. In Fig. 1, we plot the equilibrium market tightness, θ^*, as a function of the value promised by the firms to the workers, x, and conditional on the current realization of the aggregate component of productivity, y. Conditional on either realization of y, the market tightness is strictly decreasing with respect to x whenever $\theta^*(x, y)$ is positive, and it is equal to zero otherwise, which confirms the generic properties proven in Lemma 4.1. Conditional on any promised value x, the market tightness is higher when the realization of the aggregate component of productivity is higher. This property is intuitive. When y is higher, firms create more vacancies per applicant because the value of filling a vacancy is higher.[26]

[25] The data used for the calibration are described in Section 5 of Menzio and Shi [14].

[26] The reader should notice that not all submarkets need to be active in equilibrium. For example, in the non-stochastic steady state, there are only countably many active submarkets. The submarket $V_1 = m^*(U)$ is visited by the unemployed workers. The submarket $V_{n+1} = m^*(V_n)$ is visited by the workers who are employed at the wage $w_n = h(V_n)$ for

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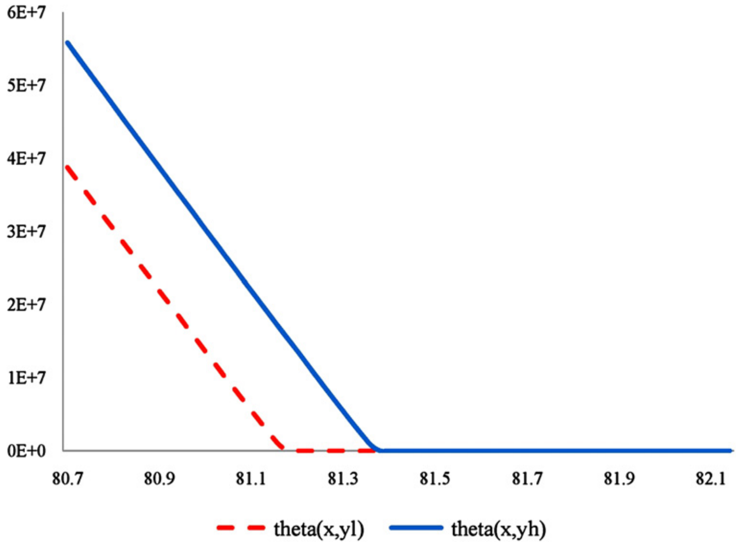

Fig. 1. Market tightness.

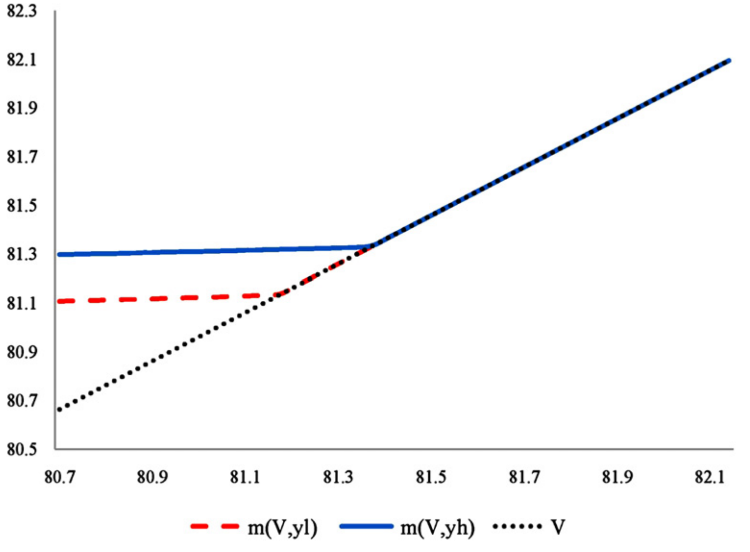

Fig. 2. Optimal search strategy.

In Fig. 2, we plot the equilibrium search strategy of a worker, m^*, as a function of the value of his current employment position, V, and conditional on the current realization of the aggregate component of productivity, y. Conditional on either realization of y, a worker chooses to look for a job that offers him the lifetime utility $m^*(V, y) > V$ whenever $V < \tilde{x}(y)$, and that offers

$n = 1, 2, \ldots$. In the stochastic equilibrium, the number of markets that are active depends on the history of realizations of the aggregate component of productivity.

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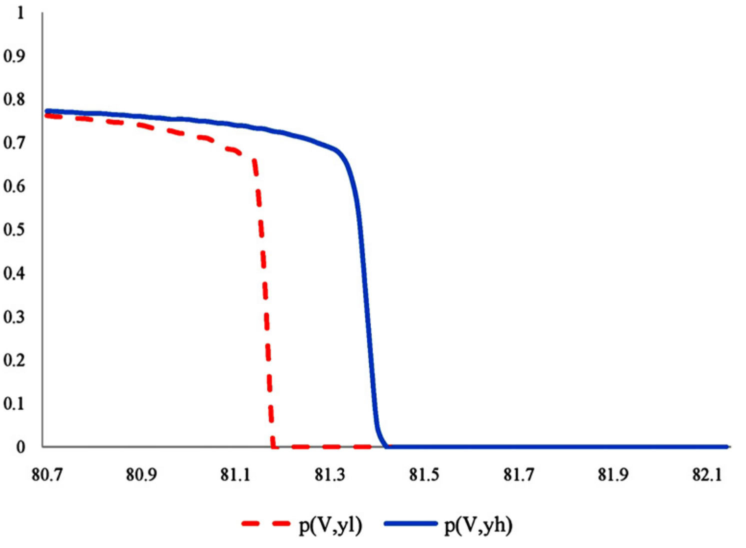

Fig. 3. Job-finding probability.

him the lifetime utility $m^*(V, y) = V$ otherwise. We proved in Section 4.2 that this property of the search strategy is generic. Conditional on any value V, a worker chooses to look for a job that offers him a higher lifetime utility when the realization of the aggregate component of productivity is higher. In Fig. 3, we plot the job-finding probability of a worker, $\tilde{p}^*$, as a function of the value of his current employment position, V, and conditional on the realization of the aggregate component of productivity, y. The probability $\tilde{p}^*$ is decreasing in V and increasing in y.

In Fig. 4, we plot the equilibrium lifetime utility of an employed worker, H^*, as a function of his wage, w, and conditional on the current realization of the aggregate component of productivity, y. Similarly, in Fig. 5, we plot the equilibrium profits of a firm that employs a worker, K^*, as a function of the wage, w, and conditional on the aggregate productivity, y.[27] Conditional on either realization of y, the lifetime utility of an employed worker is strictly increasing in w, while the profits of a firm that employs a worker are strictly decreasing in w. Conditional on any wage, both the lifetime utility of the worker and the profits of the firm are higher when the realization of the aggregate component of productivity is higher. Intuitively, when y is higher, the lifetime utility of the worker is higher because the value of searching, R^*, and the value of unemployment, U^*, are higher. The profits of the firm are higher because the amount of output produced by the worker is higher. Given these properties of K^* and H^*, it follows that the profits of a firm from filling a vacancy, J^*, are a decreasing function of the value promised to the worker, and an increasing function of the current realization of the aggregate component of productivity (see Fig. 6). Moreover, the lottery is not used in equilibrium in this example. Finally, (6.14) is satisfied everywhere along the equilibrium path and, hence, the BRE of the proxy model is also a BRE of the original model.

[27] For the sake of completeness, the reader should notice that the worker's value of unemployment, U^*, is 80.79 for $y = 0.95$, and 80.87 for $y = 1.05$.

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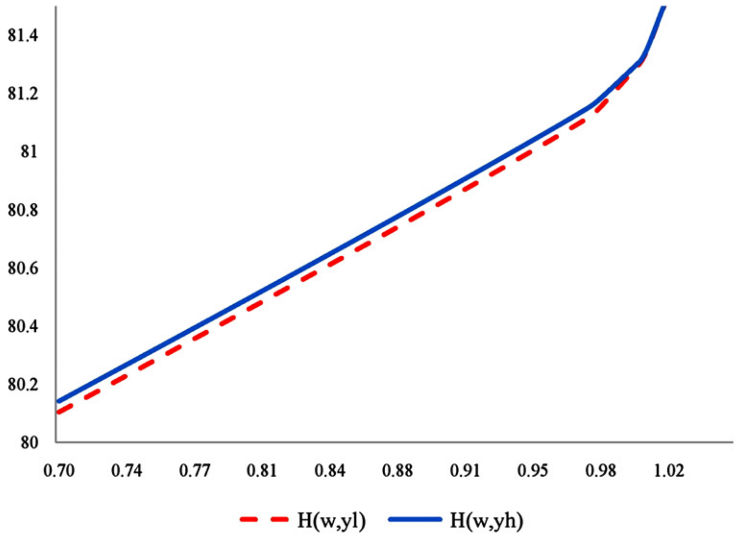

Fig. 4. Value of employment.

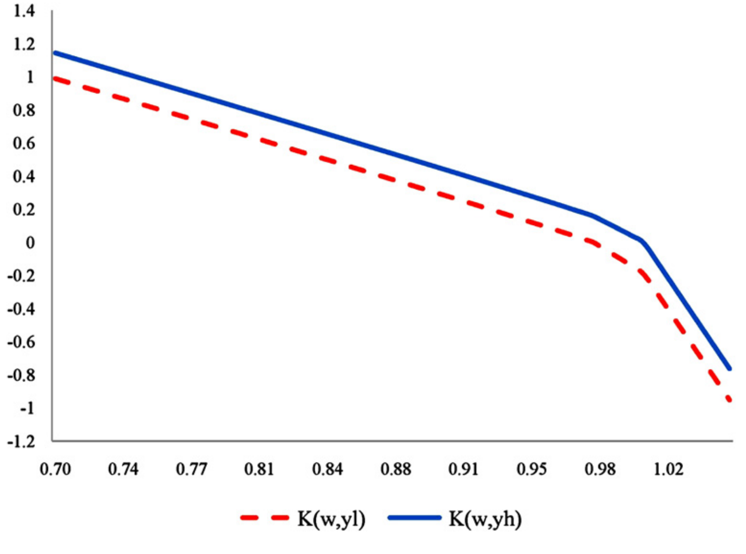

Fig. 5. Value of an employee.

By looking at Figs. 1 through 6, the reader can see that our model preserves many of the attractive features of the steady-state equilibrium of the models by Burdett and Mortensen [5], Postel-Vinay and Robin [23] and Burdett and Coles [4]. For example, since workers who have different luck with their job applications are generally employed at different wages (Figs. 2 and 4), our model generates residual wage inequality. Since workers employed at higher wages look for jobs that offer more generous terms of trade and are harder to find (Figs. 2 and 3), our model generates a positive correlation between tenure and wages. For the same reason, our model generates a negative correlation between tenure and job hazard.

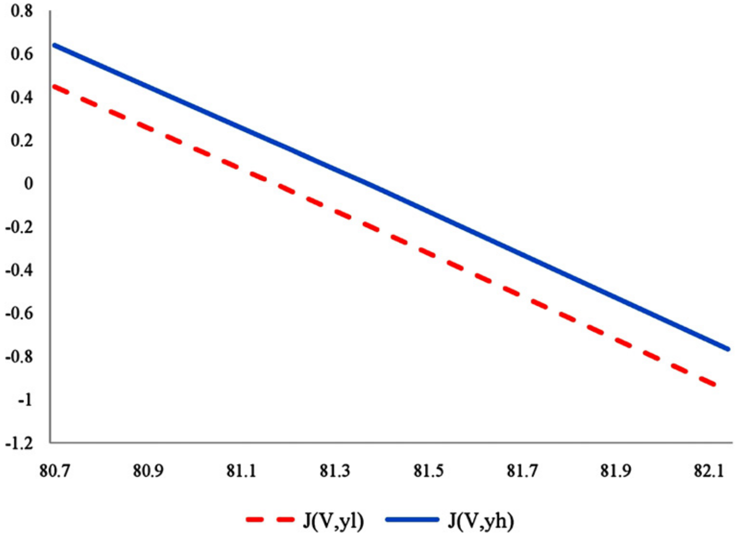

Fig. 6. Firm's value function.

By looking at Figs. 1 through 6, the reader can also see that the distribution of workers across different employment states affects the aggregate behavior of the economy even though it does not affect the agents' value and policy functions. For example, since workers employed at different jobs have different probabilities of finding a better job (Fig. 3), the distribution of workers affects the average employer-to-employer transition rate. Since workers in different employment states search in submarkets with different tightness (Figs. 1 and 3), the distribution of workers affects the vacancy rate. Finally, since workers in different jobs have different wages (Fig. 4), the distribution of workers affects the average wage. These observations also imply that the aggregate economy responds to a shock to the aggregate component of productivity through two different channels. First, the aggregate behavior of the economy is directly affected by the change in individual behavior brought about by the shock to the aggregate component of productivity. Second, the aggregate behavior of the economy is affected by the change in the distribution of workers that is brought about by the change in individuals' behavior.

8. Conclusion

In this paper, we prove existence of a BRE for a general model of directed search on the job, which allows for aggregate shocks, idiosyncratic shocks, risk aversion, and different specifications of the contractual environment. The BRE of our model preserves a number of attractive qualitative properties of the models of random search on the job by Burdett and Mortensen [5], Postel-Vinay and Robin [23], and Burdett and Coles [4]. That is, the BRE features flows of workers between employment, unemployment, and across different employers; it features residual wage inequality, and a positive return to tenure and experience. However, the BRE of our model differs from these models in that it takes into account directed search and that it is tractable for studying dynamics. In the equilibrium of the random search models, the distribution of workers across different employment states is an infinite-dimensional object which non-trivially affects the agents' value and policy functions. In the BRE of our model, the distribution of workers across different employment states does not affect the agents' value and policy functions. For this reason, while solving the equilibrium of the random search model in a stochastic environment is a difficult task both computationally and analytically, solving the BRE of our model is as easy as solving a representative agent model. These properties of the BRE make our model both a useful and a practical tool for studying labor market dynamics.

It is useful to discuss the robustness of the BRE to the introduction of ex ante heterogeneity on the worker side, since such heterogeneity is a common feature of data. As we have explained, the critical implication of directed search that supports a BRE is that workers choose to sort themselves into different submarkets. In our model, although all workers are ex ante homogeneous, their search histories induce ex post heterogeneity in the value of their employment contracts according to which the workers sort. This implication of sorting continues to hold even when workers are ex ante heterogeneous and, hence, a BRE continues to exist. To see this robustness clearly, suppose that workers are ex ante heterogeneous and the worker's type is given by $s \in S \subset \mathbb{R}^n$. For example, s may consist of the gender, age and education of the worker. A submarket is now indexed by $x : S \to X^n$, where $x(s)$ denotes the value of the employment contract offered to an applicant of type s. A submarket is now characterized by a vacancy-to-applicant ratio $\theta(x)$, and by the distribution of applicants across types, $\varphi(s, x)$. It is straightforward to verify that in equilibrium firms choose to specialize and use each offer to cater only to one particular type of workers. Because of such sorting, the distribution of applicants across types and acceptance probabilities is degenerate in every active submarket and a BRE can be shown to exist. Similar sorting with directed search has been established by [26] in an assignment model with two-sided heterogeneity and by [15] in a monetary model where buyers are heterogeneous in money holdings.

Our method for characterizing the BRE will also be useful for studying dynamics in related markets. For example, Shi [26] has used a directed search model to characterize the equilibrium and efficient patterns of the assignment between ex ante heterogeneous jobs and workers. However, he does not allow agents to continue to search after they are matched. By allowing for on-the-job search in that model, one can use the method in the current paper to study the dynamics of the assignment. Another example is the model by Gonzalez and Shi [8], who characterize a labor market equilibrium in which each unemployed worker learns about his type during search. As workers' matching histories diverge during the search process, there is a non-degenerate distribution of workers' beliefs, and this distribution is an aggregate state variable of the economy. The analysis of the equilibrium in Gonzalez and Shi [8] is tractable precisely because search is directed and the equilibrium is block recursive. However, they focus on the steady state. Using the method in the current paper, one can study aggregate dynamics of the learning equilibrium.

Appendix A. Properties of the set of functions $\mathcal{J}$

Lemma A.1. *$\mathcal{J}$ is a non-empty, bounded, closed and convex subset of the space of bounded, continuous functions on $X \times Y \times Z$, with the sup norm.*

Proof. (i) Clearly, the set $\mathcal{J}$ is non-empty and bounded.

(ii) Next, we need to prove that the set $\mathcal{J}$ is closed. To this aim, let $\{J_n\}_{n=1}^{\infty}$ be an arbitrary sequence with $J_n \in \mathcal{J}$ for every n, and with $J_n \to J$ (in the sup norm). Note that, since $J_n \to J$, for every $\varepsilon > 0$, there exists $N(\varepsilon) \geqslant 1$ such that $n \geqslant N(\varepsilon) \Rightarrow \|J_n - J\| < \varepsilon$.

For arbitrary $(y, z) \in Y \times Z$ and arbitrary $V_1, V_2 \in X$, with $V_1 \leqslant V_2$, suppose that the difference $J(V_2, y, z) - J(V_1, y, z)$ is strictly smaller than $-\bar{B}_J(V_2 - V_1)$. Let $\varepsilon > 0$ be one third of the difference between $-\bar{B}_J(V_2 - V_1)$ and $[J(V_2, y, z) - J(V_1, y, z)]$. Let n be a natural number greater than $N(\varepsilon)$. Since $\|J_n - J\| < \varepsilon$, the difference $J_n(V_2, y, z) - J_n(V_1, y, z)$ is such that

$$\begin{aligned} J_n(V_2, y, z) - J_n(V_1, y, z) &< J(V_2, y, z) - J(V_1, y, z) + 2\varepsilon \\ &= \frac{1}{3}\big[J(V_2, y, z) - J(V_1, y, z)\big] - \frac{2}{3}\bar{B}_J(V_2 - V_1) \\ &< -\bar{B}_J(V_2 - V_1). \end{aligned}$$

The last inequality contradicts $J_n \in \mathcal{J}$. Therefore, $J(V_2, y, z) - J(V_1, y, z)$ is greater than $-\bar{B}_J(V_2 - V_1)$ for all $(y, z) \in Y \times Z$ and all $V_1, V_2 \in X$, with $V_1 \leqslant V_2$. Using a similar argument, we can prove that $J(V_2, y, z) - J(V_1, y, z)$ is smaller than $-\underline{B}_J(V_2 - V_1)$ for all $(y, z) \in Y \times Z$ and all $V_1, V_2 \in X$, with $V_1 \leqslant V_2$. That is, J satisfies property (J1) of the set $\mathcal{J}$.

For all $(V, y, z) \in X \times Y \times Z$, it is immediate to verify that $J(V, y, z) \in [\underline{J}, \bar{J}]$. Hence, J satisfies property (J2) of the set $\mathcal{J}$. For arbitrary $(y, z) \in Y \times Z$, arbitrary $V_1, V_2 \in X$, and arbitrary $\alpha \in [0, 1]$, suppose that $J(V_\alpha, y, z)$ is strictly smaller than $\alpha J(V_1, y, z) + (1 - \alpha)J(V_2, y, z)$, where $V_\alpha = \alpha V_1 + (1 - \alpha)V_2$. Let $\varepsilon > 0$ be one third of the difference between $[\alpha J(V_1, y, z) + (1 - \alpha)J(V_2, y, z)]$ and $J(V_\alpha, y, z)$. Let n be a natural number greater than $N(\varepsilon)$. Since $\|J_n - J\| < \varepsilon$, we have

$$\begin{aligned} J_n(V_\alpha, y, z) &< J(V_\alpha, y, z) + \varepsilon \\ &= \alpha J(V_1, y, z) + (1 - \alpha)J(V_2, y, z) - 2\varepsilon \\ &< \alpha J_n(V_1, y, z) + (1 - \alpha)J_n(V_2, y, z) - \varepsilon \\ &< \alpha J_n(V_1, y, z) + (1 - \alpha)J_n(V_2, y, z). \end{aligned}$$

The last inequality contradicts $J_n \in \mathcal{J}$. Therefore, $J(V_\alpha, y, z)$ is greater than $\alpha J(V_1, y, z) + (1 - \alpha)J(V_2, y, z)$ for all $(y, z) \in Y \times Z$, all $V_1, V_2 \in X$ and all $\alpha \in [0, 1]$. That is, J satisfies property (J3) of the set $\mathcal{J}$. This establishes that $J \in \mathcal{J}$ and, hence, that the set $\mathcal{J}$ is closed.

(iii) Finally, we prove that the set $\mathcal{J}$ is convex. To this aim, consider arbitrary $J_1, J_2 \in \mathcal{J}$ and an arbitrary $\alpha \in [0, 1]$. Denote $J_\alpha(V, y, z) = \alpha J_1(V, y, z) + (1 - \alpha)J_2(V, y, z)$. For all $(y, z) \in Y \times Z$ and all $V_1, V_2 \in X$, with $V_1 \leqslant V_2$, the difference $J_\alpha(V_2, y, z) - J_\alpha(V_1, y, z)$ is such that

$$\begin{aligned} &J_\alpha(V_2, y, z) - J_\alpha(V_1, y, z) \\ &\quad = \alpha\big[J_1(V_2, y, z) - J_1(V_1, y, z)\big] + (1 - \alpha)\big[J_2(V_2, y, z) - J_2(V_1, y, z)\big] \\ &\quad \in \big[-\underline{B}_J(V_2 - V_1), -\bar{B}_J(V_2 - V_1)\big]. \end{aligned}$$

Hence, J_α satisfies property (J1) of the set $\mathcal{J}$. For all $(V, y, z) \in X \times Y \times Z$, it is immediate to verify that $J_\alpha(V, y, z) \in [\underline{J}, \bar{J}]$. Hence, J_α satisfies property (J2) of the set $\mathcal{J}$. For all $(y, z) \in Y \times Z$, $V_1, V_2 \in X$, and $\zeta \in [0, 1]$, let $V_\zeta = \zeta V_1 + (1 - \zeta)V_2$. Then, $J_\alpha(V_\zeta, y, z)$ is such that

$$\begin{aligned} J_\alpha(V_\zeta, y, z) &= \alpha J_1(V_\zeta, y, z) + (1 - \alpha)J_2(V_\zeta, y, z) \\ &\geqslant \alpha\big[\zeta J_1(V_1, y, z) + (1 - \zeta)J_1(V_2, y, z)\big] \\ &\quad + (1 - \alpha)\big[\zeta J_2(V_1, y, z) + (1 - \zeta)J_2(V_2, y, z)\big] \\ &= \zeta J_\alpha(V_1, y, z) + (1 - \zeta)J_\alpha(V_2, y, z). \end{aligned}$$

Hence, J_α satisfies property (J3) of the set $\mathcal{J}$. This establishes that $J_\alpha \in \mathcal{J}$ and, hence, that the set $\mathcal{J}$ is convex. □

Appendix B. Proof of Lemma 4.1

(i) For the sake of brevity, let us suppress the dependence of various functions on y and z. Let y be an arbitrary point in Y, and let x_1, x_2 be two points in X with $x_1 \leqslant x_2$. First, consider the case in which $x_1 \leqslant x_2 \leqslant \tilde{x}$. In this case, the difference $\theta(x_2) - \theta(x_1)$ is equal to

$$\theta(x_2) - \theta(x_1) = q^{-1}\big(k/J(x_2)\big) - q^{-1}\big(k/J(x_1)\big) = \int_{k/J(x_1)}^{k/J(x_2)} q^{-1\prime}(t)\,dt, \tag{B.1}$$

where the first equality uses (4.1), and the second equality uses the fact that $J(x_1) \geqslant J(x_2) \geqslant k > 0$. For all $x \in [\underline{x}, \tilde{x}]$, the derivative of $q^{-1}(.)$ evaluated at $k/J(x)$ is equal to $1/q'(\theta(x)) \in [1/q'(\bar{\theta}), 1/q'(0)]$, where $1/q'(\bar{\theta}) \leqslant 1/q'(0) < 0$. Therefore, the last term in (B.1) satisfies:

$$\frac{1}{q'(\bar{\theta})}\left(\frac{k}{J(x_2)} - \frac{k}{J(x_1)}\right) \leqslant \int_{k/J(x_1)}^{k/J(x_2)} q^{-1\prime}(t)\,dt \leqslant \frac{1}{q'(0)}\left(\frac{k}{J(x_2)} - \frac{k}{J(x_1)}\right). \tag{B.2}$$

The difference $k/J(x_2) - k/J(x_1)$ is equal to

$$\frac{k}{J(x_2)} - \frac{k}{J(x_1)} = \int_{J(x_2)}^{J(x_1)} \frac{k}{t^2}\,dt.$$

For all $x \in [\underline{x}, \tilde{x}]$, $J(x)$ is strictly decreasing in x and it is bounded between $\bar{J}$ and k. Therefore, the integral on the RHS above is such that

$$\int_{J(x_2)}^{J(x_1)} \frac{k}{t^2}\,dt \leqslant \frac{1}{k}\big[J(x_1) - J(x_2)\big] \leqslant \frac{\bar{B}_J}{k}(x_2 - x_1),$$

$$\int_{J(x_2)}^{J(x_1)} \frac{k}{t^2}\,dt \geqslant \frac{k}{\bar{J}^2}\big[J(x_1) - J(x_2)\big] \geqslant \frac{\underline{B}_J k}{\bar{J}^2}(x_2 - x_1). \tag{B.3}$$

Taken together, (B.2) and (B.3) imply that the difference $\theta(x_2) - \theta(x_1)$ is such that

$$\frac{\bar{B}_J}{q'(\bar{\theta})k}(x_2 - x_1) \leqslant \theta(x_2) - \theta(x_1) \leqslant \frac{\underline{B}_J k}{q'(0)\bar{J}^2}(x_2 - x_1). \tag{B.4}$$

Next, consider the case in which $x_1 \leqslant \tilde{x} \leqslant x_2$. Then, the difference $\theta(x_2) - \theta(x_1)$ satisfies:

$$\theta(x_2) - \theta(x_1) = \theta(\tilde{x}) - \theta(x_1) \leqslant \frac{\underline{B}_J k}{q'(0)\bar{J}^2}(\tilde{x} - x_1) \leqslant 0,$$

$$\theta(x_2) - \theta(x_1) = \theta(\tilde{x}) - \theta(x_1) \geqslant \frac{\bar{B}_J}{q'(\bar{\theta})k}(\tilde{x} - x_1) \geqslant \frac{\bar{B}_J}{q'(\bar{\theta})k}(x_2 - x_1),$$

where the first equality in both lines uses the fact that $\theta(x_2) = \theta(\tilde{x})$, and the first inequality in both lines uses the bounds in (B.4).

Finally, in the case where $\tilde{x} \leqslant x_1 \leqslant x_2$, (4.1) implies that $\theta(x_1) = \theta(x_2) = 0$.

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(ii) The function $p(\theta)$ is strictly increasing in θ, and $\theta(x)$ is strictly decreasing in x for all $x \in [\underline{x}, \tilde{x}]$. Therefore, $p(\theta(x))$ is strictly decreasing in x for $x \in [\underline{x}, \tilde{x}]$. In order to prove that the composite function $p(\theta(x))$ is strictly concave in x for $x \in [\underline{x}, \tilde{x}]$, consider arbitrary x_1, $x_2 \in [\underline{x}, \tilde{x}]$, with $x_1 \neq x_2$, and an arbitrary number $\alpha \in (0,1)$. Let $x_\alpha = \alpha x_1 + (1-\alpha)x_2$. Since the function $J(x)$ is concave in x and the function k/x is strictly convex in x, we have

$$\frac{k}{J(x_\alpha)} \leqslant \frac{k}{\alpha J(x_1) + (1-\alpha)J(x_2)} < \alpha\frac{k}{J(x_1)} + (1-\alpha)\frac{k}{J(x_2)}.$$

Since $p(q^{-1}(.))$ is strictly decreasing and weakly concave, the previous inequality implies that

$$\begin{aligned} p\big(q^{-1}\big(k/J(x_\alpha)\big)\big) &> p\big(q^{-1}\big(\alpha k/J(x_1) + (1-\alpha)k/J(x_2)\big)\big) \\ &\geqslant \alpha p\big(q^{-1}\big(k/J(x_1)\big)\big) + (1-\alpha)p\big(q^{-1}\big(k/J(x_2)\big)\big). \end{aligned} \tag{B.5}$$

Since $q^{-1}(k/J(x))$ is equal to $\theta(x)$ for all $x \in [\underline{x}, \tilde{x}]$, (B.5) can be rewritten as

$$p\big(\theta(x_\alpha)\big) > \alpha p\big(\theta(x_1)\big) + (1-\alpha)p\big(\theta(x_2)\big). \tag{B.6}$$

This establishes that $p(\theta(x))$ is strictly concave in x for all $x \in [\underline{x}, \tilde{x}]$.

Appendix C. Proof of Lemma 4.6

(i) For all $\varphi_1, \varphi_2 \in \mathcal{C}(Y)$, with $\varphi_1 \leqslant \varphi_2$, the difference $T_U\varphi_2 - T_U\varphi_1$ is such that

$$\begin{aligned} &(T_U\varphi_2)(y) - (T_U\varphi_1)(y) \\ &\quad = \beta\mathbb{E}_{\hat{\psi}}\big\{\varphi_2(\hat{y}) - \varphi_1(\hat{y}) + \lambda_u\big[\max\big\{0, R\big(\varphi_2(\hat{y}), \hat{y}\big)\big\} - \max\big\{0, R\big(\varphi_1(\hat{y}), \hat{y}\big)\big\}\big]\big\} \geqslant 0, \end{aligned} \tag{C.1}$$

where the last inequality uses the fact that the function $V + \lambda_u \max\{0, R(V)\}$ is increasing in V. For all $\varphi \in \mathcal{C}(Y)$ and all $a \geqslant 0$, $T_U(\varphi + a)$ is such that

$$\begin{aligned} \big[T_U(\varphi + a)\big](y) &= (T_U\varphi)(y) + \beta\mathbb{E}_{\hat{\psi}}\big\{a + \lambda_u\big[\max\big\{R(\varphi + a), 0\big\} - \max\big\{R(\varphi), 0\big\}\big]\big\} \\ &\leqslant (T_U\varphi)(y) + \beta a, \end{aligned} \tag{C.2}$$

where we have suppressed the dependence of various variables from the aggregate state $\hat{y}$. Conditions (C.1) and (C.2) are sufficient to prove that the operator T_U is a contraction mapping [28, Theorem 3.3]. Hence, there exists one and only one U such that $T_U U = U$.

(ii) Let $\varphi \in \mathcal{C}(Y)$ be a function that is bounded between $\underline{U}$ and $\bar{U}$. Then, $T_U\varphi$ is such that

$$\begin{aligned} &(T_U\varphi)(y) \geqslant \upsilon(b) + \beta\underline{U} = \underline{U}, \\ &(T_U\varphi)(y) \leqslant \upsilon(b) + \beta\bar{x} = \bar{U}, \end{aligned} \tag{C.3}$$

where the first line uses the facts that $\varphi \geqslant \underline{U}$ and $R(\varphi(\hat{y}), \hat{y}) \geqslant 0$; and the second line uses the fact that $\varphi + \lambda_u \max\{0, R(\varphi(\hat{y}), \hat{y})\} \leqslant \bar{x}$. From the inequalities in (C.3), it follows that the operator T_U maps the set of functions that are bounded between $\underline{U}$ and $\bar{U}$ into itself. Since the operator T_U is a contraction, it follows that its fixed point, U, is bounded between $\underline{U}$ and $\bar{U}$.

Appendix D. Proof of Lemma 6.1

(i) For all $\varphi_1, \varphi_2 \in \mathcal{C}(W \times Y)$, with $\varphi_1 \leqslant \varphi_2$, the difference $T_H\varphi_2 - T_H\varphi_1$ is such that

$$\begin{aligned}&(T_H\varphi_2)(w, y) - (T_H\varphi_1)(w, y)\\ &\quad = \beta(1-\delta)\mathbb{E}_{\hat{y}}\begin{bmatrix} \varphi_2(w,\hat{y}) + \lambda_e \max\{0, R(\varphi_2(w,\hat{y}),\hat{y})\} \\ -\varphi_1(w,\hat{y}) - \lambda_e \max\{0, R(\varphi_1(w,\hat{y}),\hat{y})\}\end{bmatrix} \geqslant 0,\end{aligned} \tag{D.1}$$

where the last inequality uses the fact that the function $V + \lambda_e \max\{0, R(V, \hat{y})\}$ is increasing in V. For all $\varphi \in \mathcal{C}(W \times Y)$ and all $a \geqslant 0$, $T_H(\varphi + a)$ is such that

$$\begin{aligned}\bigl[T_H(\varphi + a)\bigr](w, y) &= w + \beta\mathbb{E}_{\hat{y}}\bigl\{\delta U + (1-\delta)\bigl[\varphi(w,\hat{y}) + \lambda_e \max\bigl\{0, R\bigl(\varphi(w,\hat{y}),\hat{y}\bigr)\bigr\}\bigr]\bigr\}\\ &\quad + \beta(1-\delta)\mathbb{E}_{\hat{y}}\bigl\{a + \lambda_e \max\bigl\{0, R\bigl(\varphi(w,\hat{y}),\hat{y}\bigr)\bigr\}\\ &\quad - \lambda_e \max\bigl\{0, R\bigl(\varphi(w,\hat{y}) + a,\hat{y}\bigr)\bigr\}\bigr\}\\ &\leqslant (T_H\varphi)(w, y) + \beta(1-\delta)a,\end{aligned} \tag{D.2}$$

where the last inequality uses the fact that $R(V, \hat{y}) - R(V + a, \hat{y}) \leqslant 0$. Conditions (D.1) and (D.2) are sufficient to prove that the operator T_H is a contraction mapping. Hence, there exists one and only one H such that $T_H H = H$.

(ii) Let $\varphi \in \mathcal{C}(W \times Y)$ be a function that satisfies condition (6.4). Let y be an arbitrary point in Y, and w_1, w_2 two arbitrary points in W with $w_1 \leqslant w_2$. For all $\hat{y} \in Y$, the difference $f(w_2, \hat{y}) - f(w_1, \hat{y})$ is bounded between 0 and $[1 - \beta(1-\delta)]^{-1}(w_2 - w_1)$. Therefore,

$$\begin{aligned}&(T_H\varphi)(w_2, y) - (T_H\varphi)(w_1, y)\\ &\quad = w_2 - w_1 + \beta(1-\delta)\mathbb{E}_{\hat{y}}\begin{bmatrix} \varphi(w_2,\hat{y}) + \lambda_e \max\{0, R(\varphi(w_2,\hat{y}),\hat{y})\} \\ -\varphi(w_1,\hat{y}) - \lambda_e \max\{0, R(\varphi(w_1,\hat{y}),\hat{y})\}\end{bmatrix}\\ &\quad \in \bigl[1, \bigl[1 - \beta(1-\delta)\bigr]^{-1}\bigr](w_2 - w_1).\end{aligned} \tag{D.3}$$

The bounds in (D.3) imply that the operator T_H maps functions that satisfy (6.4) into functions that satisfy (6.4). Since T_H is a contraction, its unique fixed point H satisfies (6.4).

(iii) Let $\bar{V}$ denote $(\bar{w} + \beta\delta\bar{U})/[1 - \beta(1-\delta)]$. Let $\varphi \in \mathcal{C}(W \times Y)$ be an arbitrary function such that $(T_H f)(\bar{w}, y) \in [\bar{x}, \bar{V}]$ for all $y \in Y$. The function $T_H\varphi$ is such that $(T_H\varphi)(\bar{w}, y) \in [\bar{x}, \bar{V}]$ for all $y \in Y$, because

$$\begin{aligned}&(T_H\varphi)(\bar{w}, y) \geqslant \bar{w} + \beta\delta\underline{U} + \beta(1-\delta)\bar{x} = \bar{x},\\ &(T_H\varphi)(\bar{w}, y) \leqslant \bar{w} + \beta\delta\bar{U} + \beta(1-\delta)\bar{V} = \bar{V}.\end{aligned}$$

Therefore, the fixed point, H, is such that $H(\bar{w}, y) \in [\bar{x}, \bar{V}]$ for all $y \in Y$. Moreover, $H(\underline{w}, y) \leqslant \underline{x}$ for all $y \in Y$, because

$$H(\underline{w}, y) \leqslant \underline{w} + \beta\bigl[\delta\bar{U} + (1-\delta)\mathbb{E}_{\hat{y}}\bigl[\bar{V} + \lambda_e \max\bigl\{0, R(\bar{V}, \hat{y})\bigr\}\bigr]\bigr] \leqslant \underline{x}.$$

Appendix E. Proof of Lemma 6.3

(i) It is immediate to verify that, for all $\varphi_1, \varphi_2 \in \mathcal{C}(W \times Y \times Z)$, if $\varphi_1 \leqslant \varphi_2$ then $T_K\varphi_1 \leqslant T_K\varphi_2$. It is also immediate to verify that, for all $\varphi \in \mathcal{C}(W \times Y \times Z)$ and all $a > 0$, $T_K(\varphi + a)$ is smaller than $T_K\varphi + \beta a$. These two conditions are sufficient to prove that the operator T_K is a contraction mapping. Hence, there exists one and only one $K \in \mathcal{C}(W \times Y \times Z)$ such that $T_K K = K$.

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(ii)–(iii) Let $\varphi \in \mathcal{C}(W \times Y \times Z)$ satisfy conditions (6.9)–(6.10). Let (y, z) be an arbitrary point in $Y \times Z$, and w_1, w_2 arbitrary points in W with $w_1 \leqslant w_2$. The difference between $(T_K\varphi)(w_2, y, z)$ and $(T_K\varphi)(w_1, y, z)$ is such that

$$\begin{aligned}
&(T_K\varphi)(w_2, y, z) - (T_K\varphi)(w_1, y, z) \\
&\quad = w_1 - w_2 + \beta(1-\delta)\mathbb{E}_{\hat{s}}\big\{\big[1 - \lambda_e \tilde{p}\big(H(w_2, \hat{y}), \hat{y}\big)\big]\big[\varphi(w_2, \hat{y}, \hat{z}) - \varphi(w_1, \hat{y}, \hat{z})\big]\big\} \\
&\qquad + \beta(1-\delta)\mathbb{E}_{\hat{s}}\big\{\lambda_e\big[\tilde{p}\big(H(w_1, \hat{y}), \hat{y}\big) - \tilde{p}\big(H(w_2, \hat{y}), \hat{y}\big)\big]\varphi(w_1, \hat{y}, \hat{z})\big\} \\
&\quad \leqslant -\big\{1 + \beta(1-\delta)(1-\lambda_e)\underline{B}_K - \big[1 - \beta(1-\delta)\big]^{-1}\beta(1-\delta)\lambda_e \bar{B}_p \bar{K}\big\}(w_2 - w_1) \\
&\quad = -\underline{B}_K(w_2 - w_1), \qquad \text{(E.1)}
\end{aligned}$$

where the first inequality uses the bounds in (6.6), (6.9), (4.8) and (6.10). Moreover, the difference between $(T_K\varphi)(w_2, y, z)$ and $(T_K\varphi)(w_1, y, z)$ is such that

$$\begin{aligned}
&(T_K\varphi)(w_2, y, z) - (T_K\varphi)(w_1, y, z) \\
&\quad \geqslant -\big\{1 + \beta(1-\delta)\bar{B}_K - \big[1 - \beta(1-\delta)\big]^{-1}\beta(1-\delta)\lambda_e \bar{B}_p \underline{K}\big\}(w_2 - w_1) \\
&\quad = -\bar{B}_K(w_2 - w_1), \qquad \text{(E.2)}
\end{aligned}$$

where the first inequality uses the bounds (6.6), (6.9), (4.8) and (6.10).

Let w be an arbitrary point in W. Then, $T_K\varphi$ is such that

$$\begin{aligned}
&(T_K\varphi)(w, y, z) \leqslant \bar{y} + \bar{z} - \underline{w} + \beta(1-\delta)\bar{K} \leqslant \bar{K}, \\
&(T_K\varphi)(w, y, z) \geqslant \underline{y} + \underline{z} - \bar{w} + \beta(1-\delta)(1-\lambda_e)\underline{K} \geqslant \underline{K}. \qquad \text{(E.3)}
\end{aligned}$$

Inequalities (E.1)–(E.3) imply that the operator T_K maps functions that satisfy conditions (6.9)–(6.10) into functions that satisfy (6.9)–(6.10). Since the operator T_K is a contraction, its unique fixed point, K, satisfies conditions (6.9)–(6.10).

Appendix F. Two-point lotteries and concavity of the value function

Let $K(x)$ be a continuous function, where $x \in [\underline{x}, \bar{x}]$. Consider the following problem with a two-point lottery:

$$\begin{aligned}
&J(V) = \max_{(\pi, x_1, x_2)} \big[\pi K(x_1) + (1-\pi)K(x_2)\big] \\
&\text{s.t.} \quad \pi x_1 + (1-\pi)x_2 = V, \quad x_1 \leqslant V \leqslant x_2, \quad \pi \in [0, 1]. \qquad \text{(F.1)}
\end{aligned}$$

The above problem encompasses the maximization problems in (5.1) and (6.12) as special cases. (In these problems, the lottery is contingent on the realizations of aggregate and match-specific shocks, (y, z), which is suppressed here.)

To prove that $J(V)$ is concave, consider arbitrary $V \in (\underline{x}, \bar{x})$. Let (x_1^*, x_2^*) be the solution for (x_1, x_2) in (F.1). If $K(V)$ is strictly convex at V, it must be true that $x_1^* < V < x_2^*$. Thus, if $x_1^* = x_2^*$, then $J(V) = K(V)$ must be concave at V. In the remainder of the proof, it suffices to examine the case where $x_1^* < x_2^*$. For any arbitrary $x_1, x_2 \in (\underline{x}, \bar{x})$, $x_1 < x_2$, denote the line segment connecting $K(x_1)$ and $K(x_2)$ as $\overline{x_1 x_2}$, and denote the slope of $\overline{x_1 x_2}$ as

$$L(x_1, x_2) \equiv \frac{K(x_2) - K(x_1)}{x_2 - x_1}.$$

Using the constraint in (F.1) to express $\pi = (x_2 - V)/(x_2 - x_1)$, we can rewrite $J(V)$ in the following equivalent forms:

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$$J(V)=\max_{(x_1,x_2)}\big[K(x_2)-(x_2-V)L(x_1,x_2)\big]=\max_{(x_1,x_2)}\big[K(x_1)+(V-x_1)L(x_1,x_2)\big].$$

The following results hold: (A) For all $x\in[\underline{x},\bar{x}]$, $K(x)$ must lie on or below the extension of $\overline{x_1^*x_2^*}$, i.e., $K(x)\leqslant K(x_1^*)+L(x_1^*,x_2^*)(x-x_1^*)$; (B) If $x_2^*>V$, then $x_1^*=\arg\min_{x\leqslant x_2^*}L(x,x_2^*)$ and $x_2^*=\arg\max_{x\geqslant x_1^*}L(x_1^*,x)$.

Proofs of (A) and (B). For (A), consider first the case $x\in[x_1^*,x_2^*]$. (We will return to the case $x\notin[x_1^*,x_2^*]$ after proving (B).) Result (A) holds trivially when $x=x_1^*$ or $x=x_2^*$. To show that (A) also holds for $x\in(x_1^*,x_2^*)$, suppose to the contrary that (A) is violated by some $x_0\in(x_1^*,x_2^*)$. Then, $K(x_0)>K(x_1^*)+L(x_1^*,x_2^*)(x_0-x_1^*)$. If $x_0=V$, then (x_0,x_0) is optimal. If $x_0<V$, then (x_0,x_2^*) is feasible and dominates (x_1^*,x_2^*). If $x_0>V$, then (x_1^*,x_0) is feasible and dominates (x_1^*,x_2^*). The result in each of these cases contradicts the optimality of (x_1^*,x_2^*).

For (B), we only prove the first part, i.e., the part for x_1^*, since the proof of the result for x_2^* is similar. From the first rewritten form of the maximization problem, $L(x_1^*,x_2^*)\leqslant L(x,x_2^*)$ for all $x\leqslant V$. For $x\in(V,x_2^*)$, $K(x)$ is on or below the line connecting $K(x_1^*)$ and $K(x_2^*)$ (see the proven part of (A) above), and so $L(x_1^*,x_2^*)\leqslant L(x,x_2^*)$. Thus, (B) holds.

Now we prove that (A) also holds for $x\notin[x_1^*,x_2^*]$. If (A) did not hold for some $x_0<x_1^*$, then $L(x_0,x_2^*)<L(x_1^*,x_2^*)$, which would contradict (B). If (A) did not hold for some $x_0>x_2^*$, then $L(x_1^*,x_0)>L(x_1^*,x_2^*)$, which would again contradict (B). □

Lemma F.1. *$J(V)$ is a concave function.*

Proof. Let V_1 and V_2 be two arbitrary values in $[\underline{x},\bar{x}]$, and let $V_\alpha=\alpha V_1+(1-\alpha)V_2$, where $\alpha\in(0,1)$. Denote (x_{1i}^*,x_{2i}^*) as the solution to the maximization problem when $V=V_i$, where $i\in\{1,2,\alpha\}$. We show that $J(V_\alpha)\geqslant\alpha J(V_1)+(1-\alpha)J(V_2)$.

Applying (A) to any $x\in[x_{11}^*,x_{21}^*]$, we know that $K(x)$ cannot lie above the extension of $\overline{x_{1\alpha}^*x_{2\alpha}^*}$. Thus, all points on $\overline{x_{11}^*x_{21}^*}$ must lie on or below the extension of $\overline{x_{1\alpha}^*x_{2\alpha}^*}$. This implies that $J(V_1)\leqslant J(V_\alpha)-L_\alpha(V_\alpha-V_1)$, where $L_\alpha=L(x_{1\alpha}^*,x_{2\alpha}^*)$. Similarly, applying (A) to any $x\in[x_{12}^*,x_{22}^*]$ yields: $J(V_2)\leqslant J(V_\alpha)+L_\alpha(V_2-V_\alpha)$. Thus,

$$\alpha J(V_1)+(1-\alpha)J(V_2)\leqslant J(V_\alpha)+L_\alpha\big[\alpha(V_1-V_\alpha)+(1-\alpha)(V_2-V_\alpha)\big]=J(V_\alpha).$$

This completes the proof of the lemma. □

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