Directed Search on the Job, Heterogeneity, and Aggregate Fluctuations

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In models of search on the job (e.g. Kenneth Burdett and Dale Mortensen 1998, Burdett and Melvyn Coles 2003, Alain Delacroix and Shouyong Shi 2006), employed and unemployed workers search the labor market for job openings. Workers who are unemployed are willing to accept any job that makes them better off than enjoying leisure and continuing to search. Workers who are employed are willing to accept any job that offers them more than their current job does. On the other side of the market, firms are indifferent between opening jobs that offer different wages, as firms that offer higher wages can fill their jobs faster and retain their workers for a longer period of time. The extent of search frictions determines how quickly workers move up in the wage-offer distribution and how the wage-offer distribution itself looks like. Overall, models of search on the job provide an equilibrium theory of workers' transitions between employment, unemployment and across employers, and, simultaneously, a theory of wage inequality. Because of these properties, models of search on the job are a useful tool for studying the labor market.

Under the standard assumption of random search, these models are difficult to solve outside of the steady state because agents need to forecast the evolution of the infinite-dimensional wage distribution in order to solve their individual problems. For example, firms need to forecast the evolution of the wage distribution in order to compute the probability of filling a vacancy. Similarly, workers need to compute the evolution of the wage distribution in order to compare the value of a particular job offer with the value of their current employment position. Because of this technical property, models of random search on the job are difficult to use for studying the aggregate dynamics of the labor market. For example, these models are difficult to use for studying the response of unemployment, vacancies and wages to aggregate productivity shocks, or for studying the welfare effects of a change in labor market policies.

However, when the standard assumption of random search is replaced with the assumption of directed search, we (in Guido Menzio and Shi 2009, and Menzio and Shi forthcoming) show that models of search on the job are easy to solve even outside of the steady state and, hence, can be used for studying the aggregate dynamics of the labor market. Specifically, building on Shi (2009), we develop a general model of directed search on the job, which allows for aggregate shocks, idiosyncratic shocks, and different specifications of the contractual environment. For this model, we formally establish existence of an equilibrium in which agents do not need to forecast the evolution of the wage distribution in order to solve their individual problems. Instead, agents only need to forecast the evolution of the exogenous aggregate shocks. We refer to this equilibrium as a Block Recursive Equilibrium (BRE). Solving for the BRE of our model is as easy as solving the equilibrium of a representative-agent model, in and out of the steady-state. Moreover, the BRE of our model preserves all the desirable properties of traditional models of search on the job regarding workers' transitions and wage inequality.

In this paper, we present our model of directed search on the job. We outline the proof of existence of a BRE. We explain why a BRE does exist under the assumption of directed search and why it does not under the assumption of random search. Finally, we generalize existence of a BRE to a version of the model in which workers are allowed to be ex-ante heterogeneous with respect to some observable characteristics such as education and skill.

I. Model

We consider an economy populated by a continuum of workers with measure 1, and by a continuum

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of firms with positive measure. Each worker is endowed with an indivisible unit of labor. Each worker maximizes the expected sum of periodical utilities $\sum_{t=0}^{\infty}\beta^t \upsilon(c_t)$, where β is the discount factor and υ is the periodical utility function with $\upsilon' \in [\underline{\upsilon}', \overline{\upsilon}']$, $\underline{\upsilon}' > 0$, and $\upsilon'' \leq 0$. Each firm operates a constant return to scale technology that turns 1 unit of labor into $y + z$ units of output. The first component of productivity, y, is common to all firms, and its value lies in the set $Y = \{y_1, y_2, ..., y_{N(y)}\}$, where $y_1 < y_2 < ...y_{N(y)}$ The second component of productivity, z, is specific to a firm-worker pair, and its value lies in the set $Z = \{z_1, z_2, ..., z_{N(z)}\}$, where $z_1 < z_2 < ... < z_{N(z)}$. Each firm maximizes the expected sum of periodical profits discounted at the factor β.

The labor market is organized in a continuum of submarkets indexed by the expected lifetime utility x that the firms offer to the workers, $x \in [\underline{x}, \overline{x}] = X$, with $x < \upsilon(b)/(1-\beta)$ and $\overline{x} > \upsilon(y_{N(y)} + z_{N(z)})/(1-\beta)$. Specifically, whenever a firm meets a worker in submarket x, the firm offers the worker an employment contract that gives him the expected lifetime utility x. In submarket x, the ratio of the number of vacancies created by firms to the number of workers looking for jobs is given by the tightness $\theta(x) \geq 0$ which is determined in equilibrium.

At the beginning of each period, the state of the economy can be summarized by the triple $(y, u, g) = \psi$. The first element of ψ is the aggregate component of productivity. The second element is the measure of unemployed workers, $u \in [0, 1]$. The third element is a function $g : X \times Z \to [0, 1]$, with $g(V, z)$ denoting the measure of workers who are employed at jobs that give them the lifetime utility $\tilde{V} \leq V$ and that have an idiosyncratic component of productivity $\tilde{z} \leq z$.

Each period is divided into four stages: separation, search, matching and production. At the separation stage, an employed worker is forced to move into unemployment with probability $\delta \in (0, 1]$. Also, at the separation stage, an employed worker has the option to voluntarily move into unemployment. At the search stage, a worker (employed or unemployed) chooses in which submarket to look for a job, and a firm chooses how many vacancies to create and where to locate them. The cost of maintaining a vacancy is $k > 0$. Both workers and firms take $\theta(x)$ parametrically.

At the matching stage, the workers and the vacancies in submarket x come together through a frictional matching process. In particular, a worker meets a vacancy with probability $p(\theta(x))$, where p is a strictly increasing and concave function such that $p(0) = 0$ and $p'(0) > 0$. Similarly, a vacancy meets a worker with probability $q(\theta(x))$, where q is a strictly decreasing and convex function such that $q(\theta) = p(\theta)/\theta$, $q(0) = 1$, $q'(0) < 0$, and $p(q^{-1}(.))$ concave. When a vacancy and a worker meet, the firm that owns the vacancy offers to the worker an employment contract that gives him the lifetime utility x. If the worker rejects the offer, he returns to his previous employment position. If the worker accepts the offer, the two parties form a new match with idiosyncratic productivity z_0.

At the production stage, an unemployed worker produces and consumes $b > 0$ units of output. A worker employed at a job z produces $y + z$ units of output and consumes w of them, where w is specified by the worker's labor contract. At the end of the production stage, Nature draws next period's aggregate component of productivity, $\hat{y}$, from the probability distribution $\Phi_y(\hat{y}|y)$, and next period's idiosyncratic component of productivity, $\hat{z}$, from the distribution $\Phi_z(\hat{z}|z)$. The draws of the idiosyncratic component of productivity are independent across matches.

We consider two alternative contractual environments. In the "dynamic contracts" environment, a firm commits to an employment contract that specifies the worker's wage as a function of the history of realizations of the idiosyncratic productivity of the match, z, the history of realizations of the aggregate state of the economy, ψ, and the history of realizations of a lottery that is drawn at the beginning of every production stage. This environment generalizes the contractual environment considered by Burdett and Coles (2003) and Shi (2009) to an economy with stochastic productivity. In the "fixed-wage contracts" environment, a firm commits to a wage that remains constant throughout the entire duration of the employment relationship. This constant wage is allowed to depend only on the outcome of a lottery that is drawn at the beginning of the employment relationship. This environment is similar to the one considered by Burdett and Mortensen (1998).

Notice that, in general, the contracting problem of the firm is a complicated sequence problem in which the history upon which wages are contingent grows to infinity over time. However, following the literature on dynamic contracts, we can rewrite this problem recursively by using the worker's lifetime utility as an auxiliary state variable.

II. Block Recursive Equilibrium

We denote as $U(y)$ the lifetime utility of a worker who is unemployed, given that the aggregate component of productivity is y. We denote as $J(V, y, z)$ the profits of a firm that employs a worker, given that the employment contract is worth the lifetime utility V to the worker, and the idiosyncratic component of the firm-worker match is z. The value functions U and J are measured at the production stage. We denote as $R(V, y)$ the lifetime utility of a worker at the search stage, given that the worker's current employment position is worth V. We denote as $\theta(x, y)$ the equilibrium tightness of submarket x. Finally, we denote with $\Phi_\psi(\hat{\psi}|\psi)$ the transition probability function for the aggregate state of the economy ψ.

Now, we are in the position to define a BRE for the environment with dynamic contracts. The reader can find the definition of a BRE for the environment with fixed-wage contracts in Menzio and Shi (forthcoming).

DEFINITION 1: *A BRE is a list of value and policy functions* (θ, R, m, U, J, c) *together with a transition probability function* Φ_ψ. *These functions satisfy the following conditions:*
(i) For all $(V, \psi) \in X \times \Psi$,

$$R(V, y) = V + \max_{x \in X} \{p(\theta(x, y))(x - V)\},$$

and m *is the associated policy function;*
(ii) For all $\psi \in \Psi$,

$$U(y) = \upsilon(b) + \beta \mathbb{E} R(U(\hat{y}), \hat{y});$$

(iii) For all $(V, \psi, z) \in X \times \Psi \times Z$,

$$J(V, y, z) = \max_{w, \hat{V} \in X} y + z - w + \beta \mathbb{E}\left[\begin{array}{c}(1 - d(\hat{y}, \hat{z}))\times \\ (1 - \tilde{p}(\hat{y}, \hat{z})) J(\hat{V}(\hat{y}, \hat{z}), \hat{y}, \hat{z})\end{array}\right],$$

subject to the constraints

$$V = \upsilon(w) + \beta \mathbb{E}\left[\begin{array}{c} d(\hat{y}, \hat{z}) R(U(\hat{y}), \hat{y}) \\ +(1 - d(\hat{y}, \hat{z}))\hat{V}(\hat{y}, \hat{z})\end{array}\right],$$

$$d(\hat{y}, \hat{z}) = \{1 \text{ if } U(y) > V(y, z),\ \delta \text{ else}\},$$

where $\tilde{p}(\hat{y}, \hat{z}) \equiv p(\theta(m(\hat{V}(\hat{y}, \hat{z})), \hat{y}))$ *and* c *is the associated policy function;*
(iv) For all $(x, \psi) \in X \times \Psi$,

$$k \geq q(\theta(x, y)) J(x, y, z_0)$$

and $\theta(x, y) \geq 0$, *with complementary slackness;*
(v) For all $\psi \in \Psi$, Φ_ψ *is consistent with the transition probability of the exogenous variables,* y *and* z, *and with the policy functions* m *and* c.

Conditions (i)-(v) are straightforward to interpret. Condition (i) insures that, given the tightness of each submarket x, the worker chooses where to apply for a job to maximize his lifetime utility. Condition (iii) insures that, given the optimal search strategy of the worker, the firm chooses the employment contract to maximize its profits and to provide the worker with the promised lifetime utility V. Condition (iv) insures that the number of vacancies created in each submarket x maximizes the profits of the firm. Finally, condition (v) insures that the law of motion for the aggregate state of the economy is consistent with the stochastic process for productivity and the agents' policy functions. Overall, conditions (i)-(v) insure that, in a BRE just like in a standard recursive equilibrium, the choices of each agent are optimal given the other agents' choices.

However, unlike in a standard recursive equilibrium, in a BRE the agents' value and policy functions (θ, R, m, U, J, c) depend on the aggregate state of the economy only through the aggregate component of productivity, y, and not through the whole distribution of workers across different employment states, (u, g). Intuitively, this means that in a BRE agents do not need to forecast the evolution of the entire distribution of workers to solve their problem, but only the evolution of their individual state and of the aggregate component of productivity. Formally, this means that, in order to solve for a BRE, one needs to solve a system of functional equations in which the unknown functions depend on three real numbers (the individual state variables, V and z, and the driving force of aggregate uctuations, y) and not on an entire function (the distribution of workers across employment states). Solving such a system of functional equation is as easy as solving the equilibrium of a representative agent model. We refer to the equilibrium as a Block Recursive Equilibrium because it is a recursive equilibrium and the block of equilibrium conditions that describe the agents' value and policy functions can be solved before having solved the law of motion of the aggregate state of the economy.

III. Existence

While we can define a Block Recursive Equilibrium for any model of search on the job, there is no reason to believe that such an equilibrium would generally exist. In fact, for models of random search on the job, it is easy to establish that no equilibrium is block recursive. However, for our model of directed search on the job, we can establish the following result.

THEOREM 2: *A Block Recursive Equilibrium exists.*

The proof of Theorem 2 is in Menzio and Shi (forthcoming). Here, we present a sketch of the proof. Let $\mathcal{J}$ denote the set of bounded and continuous profit functions that depend on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y, and not through the distribution of workers across employment states, (u, g). We take an arbitrary profit function J from the set $\mathcal{J}$. Given J, the market tightness function, θ, that solves the equilibrium condition (iv) depends on ψ only though y and not through (u, g). Intuitively, since the value to a firm of filling a vacancy in submarket x does not depend on the distribution of workers across employment states and the cost of creating a vacancy is constant, the probability that a firm fills a vacancy in submarket x, and hence the tightness of submarket x, must be independent of (u, g).

Given θ, the search value function, R, that solves the equilibrium condition (i) depends on ψ only through y. Intuitively, R does not depend on (u, g) because neither the probability that a worker meets a vacancy in submarket x nor the benefit to a worker of meeting a vacancy in submarket x depends on the distribution of workers across employment states. Given R, the unemployment value function, U, that solves the equilibrium condition (ii) depends only on ψ only through y. Intuitively, U does not depend on (u, g) because neither the output of an unemployed worker nor his return to search depends on the distribution of workers across employment states.

After inserting J, θ, R and U into the RHS of the equilibrium condition (iii), we obtain an update for the profit function, TJ. The profit function TJ depends on ψ only through y. Intuitively, TJ does not depend on (u, g), because the output of the match in the current period, the probability that the match survives until the next production stage, and the value to the firm of the match at the next production stage are all independent of the distribution of workers. Hence, the equilibrium operator T maps the set of profit functions $\mathcal{J}$ into itself. Moreover, the operator T satisfies the assumptions of Schauder's fixed point theorem, and, hence, there exists a $J^* \in \mathcal{J}$ such that $J^* = TJ^*$. Given J^*, we construct the value and policy functions θ^*, R^*, m^*, U^* as described above. Taken together, the functions $(\theta^*, R^*, m^*, U^*, J^*, c^*)$ satisfy the equilibrium conditions (i)-(iv) and depend on the aggregate state of the economy ψ only through y. Hence, they constitute a BRE.

Directed search is necessary for existence of a BRE. To see this necessity clearly, suppose that search is random, instead. Then the equilibrium condition (iv) is replaced by

$$(1) \qquad k \geq \max_{x \in X} q(\theta(\psi)) \mathbb{I}(x, \psi)\, J(x, \psi, z_0),$$

and $\theta(\psi) \geq 0$, with complementary slackness. The term on the LHS of (1) is the cost of creating a vacancy. The expression on the RHS of (1) is the maximized benefit of creating a vacancy. The first term on the RHS is the probability that a firm meets a worker. The second term denotes the probability that a randomly selected worker is willing to accept an employment contract that offers him the value x. The third term is the value to a firm of having an employee to whom it has promised the value x. Notice that the probability that a randomly selected worker is willing to accept a contract that offers x depends on the distribution of workers across employment states. That is, $\mathbb{I}(x, \psi)$ depends on ψ not only through y, but also through (u, g). In turn, this implies that the equilibrium condition (1) holds only if the distribution affects also the market tightness θ or the profit function J. In either case, the equilibrium fails to be block recursive.

In contrast, directed search eliminates the dependence of the acceptance probability on the distribution of workers because a worker always accepts a job that he chooses to search for. That is, $\mathbb{I}(x, \psi) = 1$. In turn, $\mathbb{I}(x, \psi) = 1$ guarantees that, if the profit function J does not depend on the distribution of workers, neither do θ, R, m, U, c and the updated profit function TJ.

IV. Ex-ante Heterogeneous Workers

For some empirical applications, it might be necessary to use a version of the model in which workers are ex-ante heterogeneous in some observable characteristics such as education or skill. In this section, we establish existence of a BRE for such a version of the model.

We consider an economy populated by N types of workers. Specifically, there is a continuum of workers of type s_i with measure π_i, $i = 1, 2, ..., N$, where $s_1 < s_2 < ... < s_N$ and $\sum_{i=1}^{N} \pi_i = 1$. A worker's type affects his labor productivity. Specifically, a worker of type s_i produces $y + z + s_i$ units of output when employed in a match with idiosyncratic productivity z. In this economy, the labor market is organized in a continuum of submarkets indexed by the vector $\mathbf{x} \in X^N$. In submarket $\mathbf{x}$, firms offer the expected lifetime utility $\mathbf{x}_i$ to workers of type s_i, where $\mathbf{x}_i$ denotes the i-th component of $\mathbf{x}$. In submarket $\mathbf{x}$, the ratio of the number of vacancies created by firms to the number of workers looking for jobs is given by $\theta(\mathbf{x}) \geq 0$, and the distribution of workers across types is given by $\phi(\mathbf{x})$.

We denote as $U(s_i, y)$, $R(s_i, V, y)$ and $m(s_i, V, y)$ the value and policy functions of a worker of type s_i, and with $m_j(s_i, V, y)$ the j-th component of $m(s_i, V, y)$. Similarly, we denote as $J(s_i, V, z, y)$ and $c(s_i, V, z, y)$ the value and policy function of a firm that employs a worker of type s_i.

THEOREM 3: *Let* $(\theta^i, R^i, m^i, U^i, J^i, c^i)$ *be a BRE of the economy with ex-ante homogeneous workers of type* s_i. *The economy with ex-ante heterogeneous workers admits a BRE with the following properties for* $i = 1, 2, ..., N$*: (i) For all* $(V, y) \in X \times Y$, $R(s_i, V, y) = R^i(V, y)$ *and* $m_i(s_i, V, y) = m^i(V, y)$*; (ii) For all* $y \in Y$, $U(s_i, y) = U^i(y)$*; (iii) For all* $(V, z, y) \in X \times Z \times Y$, $J(s_i, V, z, y) = J^i(V, z, y)$ *and* $c(s_i, V, z, y) = c^i(V, z, y)$*; (iv) For all* $(V, y) \in X \times Y$, $\theta(m(s_i, V, y)) = \theta^i(m^i(V, y))$.

The proof of Theorem 3 is in the online Appendix. The theorem states that there exists a BRE of the economy with ex-ante heterogeneous workers that is given by the stratification of the BRE of the N economies with ex-ante homogeneous workers of type s_i, $i = 1, 2, ..., N$. Specifically, in the BRE of the economy with ex-ante heterogeneous workers, the value and policy functions of a worker of type s_i are the same as in the BRE of an economy in which all workers are of type s_i. Similarly, the value and policy functions of a firm that employs a worker of type s_i are the same as in the BRE of the economy with homogeneous workers of type s_i. Practically, Theorem 3 implies that, in order to solve for the BRE of the economy with ex-ante heterogeneity, it is sufficient to solve for the BRE of the N economies with ex-ante identical workers.

Let us give some intuition for Theorem 3. In the economy with ex-ante heterogeneous workers, the labor market is endogenously segmented by worker's type, in the sense that any active submarket is visited exclusively by one type of worker. Given that the labor market is segmented, it is easy to understand why there exists a BRE given by the stratification of the BRE of the N economies with ex-ante homogeneous workers. Now, to understand why the labor market is segmented, consider the following thought experiment. Suppose that submarket $\mathbf{x} = (x_1, x_2)$ is visited by workers of type s_1 and s_2. Without loss in generality, suppose that the profits of the firm are greater if it fills a vacancy in submarket $\mathbf{x}$ with a worker of type s_1 rather than s_2. Then, in submarket $\mathbf{x}' = (x_1, \underline{x})$, the expected profits of the firm from filling a vacancy are greater than in submarket $\mathbf{x}$, because no worker of type s_2 searches in $\mathbf{x}'$. Hence, the probability that a firm fills a vacancy must be lower in submarket $\mathbf{x}'$ than in submarket $\mathbf{x}$ and, in turn, $\theta(\mathbf{x}')$ must be greater than $\theta(\mathbf{x})$. But if $\theta(\mathbf{x}')$ is greater than $\theta(\mathbf{x})$, a worker of type s_1 is better off applying for a job in submarket $\mathbf{x}'$ than in $\mathbf{x}$. This contradicts the conjecture that submarket $\mathbf{x}$ is visited by both types of workers.

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Proof of Theorem 3

In order to keep the exposition as simple as possible, we prove the theorem for $N = 2$. The proof for $N \geq 3$ is similar.

Theorem 2 guarantees that there exists at least one BRE for the economy with ex-ante homogeneous workers of type s_i, $i = 1, 2$. Let $(\theta^i, R^i, m^i, U^i, J^i, c^i)$ be any BRE for the economy with ex-ante homogeneous types of type s_i. Let $(\theta, \phi, R, m, U, J, c)$ denote a candidate BRE for the economy with ex-ante heterogeneous agents. Fix an arbitrary $y \in Y$ and, without loss in generality, suppose that $\theta_1(\underline{x}, y) \geq \theta_2(\underline{x}, y)$. Now, choose the market tightness function, θ, and the distribution of applicants, ϕ, as follows. For all $(x_1, x_2) \in X \times (\underline{x}, \overline{x}]$, let $\theta(x_1, x_2) = \min\{\theta^1(x_1), \theta^2(x_2)\}$ and let $\phi(x_1, x_2) = (1, 0)$ if $\theta^1(x_1) < \theta^2(x_2)$ and $\phi(x_1, x_2) = (0, 1)$ if $\theta^1(x_1) > \theta^2(x_2)$. For all $x_1 \in X$, let $\theta(x_1, \underline{x}) = \theta^1(x_1)$ and $\phi(x_1, \underline{x}) = (1, 0)$. That is, for $x_2 > \underline{x}$, we set the tightness of submarket (x_1, x_2) to the minimum between the tightness of submarket x_1 in an economy with ex-ante homogeneous workers of type 1, and the tightness of submarket x_2 in an economy with ex-ante homogeneous workers of type 2. For $x_2 = \underline{x}$, we set the tightness of submarket (x_1, x_2) to be the tightness of submarket x_1 in an economy with ex-ante heomogeneous workers of type 1. Notice that, in the previous expressions, we have omitted the dependence of various functions on y. We shall do the same in the remainder of the proof.

Next, choose the search value function, R, the search policy function, m, the profit function, J, and the unemployment value function, U, as follows. For all $V \in X$ and $i = 1, 2$, let $R(s_i, V) = R^i(V)$. For all $V \in X$, let $m(s_1, V) = (m^1(V), \underline{x})$ and $m(s_2, V) = (\underline{x}, m^2(V))$. For all $(V, z) \in X \times Z$ and $i = 1, 2$, let $J(s_i, V, z) = J^i(V, z)$ and $c(s_i, V, z) = c^i(V, z)$. For $i = 1, 2$, let $U(s_i, y) = U^i(y)$. In words, the lifetime utility of worker s_i in an economy with ex-ante heterogeneous workers is set equal to the lifetime utility of a worker in an economy with ex-ante homogeneous workers of type s_i. Similarly, the profits of a firm from employing a worker s_i in an economy with ex-ante heterogeneous workers are equal to the profits of a firm in an economy with ex-ante homogeneous workers of type s_i.

Now, we verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium conditions (i)-(iv) and, hence, it is a BRE for the economy with ex-ante heteorogeneous workers. First, we verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium condition (iv). Consider a submarket $(x_1, x_2) \in X^2$ such that either $\theta^1(x_1) \leq \theta^2(x_2)$ or $x_2 = \underline{x}$. In this case, we have

$$\begin{aligned} &q(\theta(x_1, x_2)) \textstyle\sum_{i=1}^{2} \phi_i(x_1, x_2) J(s_i, x_i, z_0) = q(\theta^1(x_1)) J^1(x_1, z_0) \leq k, \\ &\theta(x_1, x_2) = \theta^1(x_1) \geq 0, \end{aligned} \tag{2}$$

with complementary slackness. The first line in (2) denotes as $\phi_i(x_1, x_2)$ the i-th component of the vector $\phi(x_1, x_2)$ and makes use of the equations $\phi(x_1, x_2) = (1, 0)$, $J(s_i, x_i, z_0) = J^i(x_i, z_0)$, the second line makes use of the equation $\theta(x_1, x_2) = \theta^1(x_1)$, and both lines use the fact that $(\theta^1, R^1, m^1, U^1, J^1, c^1)$ is a BRE. The inequalities in (2) imply that the equilibrium condition (iv) is satisfied for all $(x_1, x_2) \in X^2$ such that either $\theta^1(x_1) \leq \theta^2(x_2)$ or $x_2 = \underline{x}$. Using a similar argument, we can prove that the equilibrium condition (iv) is satisfied for all other submarkets.

Next, we verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium condition (i). Consider an arbitrary $x_1 \in X$. For all $x_2 \in (\underline{x}, \overline{x}]$, the tightness of submarket (x_1, x_2) is $\theta(x_1, x_2) \leq \min\{\theta^1(x_1), \theta^2(x_2)\}$. For $x_2 = \underline{x}$, the tightness of submarket (x_1, x_2) is $\theta(x_1, x_2) = \theta^1(x_1)$. Since these results hold for an arbitrary x_1, we have that

$$\begin{aligned} &\max_{(x_1, x_2) \in X^2} p(\theta(x_1, x_2))\,(x_1 - V) \\ &= \max_{x_1 \in X} p(\theta^1(x_1))(x_1 - V) \\ &= R^1(V) = R(s_1, V), \end{aligned} \tag{3}$$

where the third line makes use of the fact that $(\theta^1, R^1, m^1, U^1, J^1, c^1)$ is a BRE. Moreover, we have that

$$p(\theta(m(s_1, V)))\,(m_1(s_1, V) - V) = p(\theta^1(m^1(V)))(m^1(V) - V), \tag{4}$$

where $m_1(s_1, V)$ denotes the first component of the vector $m(s_1, V)$. Taken together, equations (3) and (4) imply that the equilibrium condition (i) is satisfied for all $V \in X$ and $i = 1$. Using a similar argument, we can prove that the equilibrium condition (i) is satisfied also for $i = 2$. Moreover, notice that the distribution of applicants ϕ across types is consistent with the worker's equilibrium search strategy m.

Finally, it is straightforward to verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium conditions (ii) and (iii). Hence, $(\theta, \phi, R, m, U, J, c)$ is a BRE for the economy with ex-ante heterogeneous agents.