A Cheap-Talk Theory of Random and Directed Search

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Abstract

This paper studies a search model of the labor market where firms have private information about the gains from trade and post cheap-talk messages to advertise their vacancies before workers decide which location to visit. The surplus of a match is divided ex-post according to the outcome of an alternating-offer bargaining game of asymmetric information. When this bargaining game is fast, I show that the maximum amount of information that can be transmitted through the cheap-talk depends non-monotonically on the tightness of the labor market. In particular, if the ratio of unemployed workers to vacancies is either sufficiently high or sufficiently low, the unique equilibrium has the firms babbling and search is random. If the tightness of the labor market takes on intermediate values, there is also an equilibrium where the cheap-talk is informative, high and low productivity firms post different messages and the search process of the workers is directed.

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1 Introduction

Imagine a search-theoretic model of the labor market where heterogeneous firms and workers have private information about their own characteristics -such as skills supplied or needed, talent, productivity of the open vacancies, etc.. . . - and the communication technology allows them to costlessly and publicly advertise their type in the marketplace. Also, imagine that the legal framework is such that seeking retribution in court from an agent that has misrepresented herself in an advertisement is not economically feasible.

In this framework, firms and workers choose which message to post in the marketplace based on how it affects the probability of meeting a trading partner (or a certain type of partner) and how it affects the division of the surplus once a match is created. When different types have an incentive to post different messages, given that the messages are correctly interpreted by the market, then there exists an equilibrium of the economy where advertisement is informative and it directs the search process. When strategic information transmission cannot be sustained in the communication game, then agents correctly ignore the advertisements and the search process is random.

In this paper, I consider a simple version of the model economy described above and provide a characterization of the relationship between the fundamentals of the economy –such as the tightness of the labor market and the distribution of the bargaining power– and the informativeness of communication. More specifically, I consider a labor market populated by n homogeneous workers that are looking for work and by m heterogeneous firms that are looking for applicants to fill their vacancy. In the first stage of the game, each firm publishes a cheap-talk message about the productivity of its job opening, a variable that is idiosyncratic and privately observed by the firm. In the second stage of the game, after having observed the distribution of posted messages, each worker chooses which vacancy to visit. In the final stage of the game, each firm randomly chooses one of its applicants –given that somebody has applied to the firm– and the two parties enter a "fast" alternating-offer bargaining game under asymmetric information.

The central finding of the paper is that the maximum amount of information that can be transmitted in equilibrium depends non-monotonically on the tightness of the labor market. When unemployed workers are sufficiently scarce with respect to vacancies, any message that advertises a vacancy as "high productivity" is not credible, the unique equilibrium of the cheap-talk game is "babbling" and workers search at random. When vacancies are sufficiently scarce with respect to unemployed workers, no firm has an incentive to separate from the low type, communication is not informative and search is random. When the tightness m/n of the labor market takes on an intermediate value, there exists an equilibrium where the firm's type space is partitioned into two connected intervals. All the firms whose productivity is lower than an endogenous cutoff point post a "low-quality vacancy" message, while higher types post "high-quality vacancy." In this equilibrium, workers visit more frequently the high-quality vacancies, because they (rationally) expect a higher wage to be the outcome of the bargaining game.

In order to understand the non-monotonic relationship between information and market tightness and the qualitative properties of the separating equilibrium, it is necessary to proceed backwards form the characterization of the bargaining game as a function of worker's beliefs to the equilibria of the cheap-talk game. As first conjectured by Coase and then proved by Gul and Sonnenschein (1988), for any sequential equilibrium of the bargaining game of asymmetric information that satisfies some mild regularity conditions, all potential gains from trade are realized within ϵ time when the time delay between offers converges to zero. Moreover, when the off-equilibrium conjectures are restricted in the spirit of Grossman and Perry (1986b), I can prove that the limiting outcome of the bargaining game coincides with the outcome of the perfect information version of the game played by the worker and the lowest firm's type on the support of worker's beliefs (conditional on the message posted by the firm at the communication stage).

Because the bargaining game ends instantly and the terms of trade are independent on the firms' type, from a partial equilibrium perspective, every message is associated to some promised wage. If a firm posts a "high wage" rather than a "low wage" message, it will receive on average more applicants, it will fill its vacancy with higher probability and it will concede to pay a higher wage to its employee. Since this payoff differential is increasing in the firm's productivity, it follows that in any equilibrium of the cheap-talk game the type space is divided into connected subsets, where subsets with lower productivity post messages associated to lower wages.

In general equilibrium, the wage attached to each message depends on the subset of firms that uses that message to advertise their vacancies. In particular, the wage is a linear function of the productivity of the lowest type within the subset of firms that post the message. Therefore, a separating equilibrium of the cheap-talk game exists if and only if there is a type that is just indifferent between posting the "lowest wage" message and credibly revealing its productivity to the market. On the one hand, the cost of revealing the true productivity rather than posting the "lowest wage" message is a linear function of the firm's type, with a slope that depends on the relative bargaining power of the two parties. On the other hand, because the matching process is such that the probability of filling a vacancy increases at decreasing rates with the wage, the net gain from revealing the true productivity is an increasing and concave function of the firm's type, with a slope that decreases with the ratio of workers to vacancies. When unemployed workers are abundant, the cost of transmitting information dominates the gain for every possible cutoff type and "babbling" is the only equilibrium of the cheap-talk game. When vacancies are abundant, for every type, the gain of revealing the true productivity dominates the cost and the cheap-talk is uninformative. For intermediate levels of market tightness, there exists an indifferent type and a separating equilibrium of the cheap-talk game.

The second finding of the paper is that information has ambiguous effects on efficiency. On the one hand, in a directed search equilibrium workers visit more frequently high productivity firms and the average output of a match is higher than in a random search equilibrium. On the other hand, in a directed search equilibrium fewer matches are created, because the probability of filling a vacancy is a concave function of the queue length. The magnitude of the positive composition effect created by information depends on the difference in the average productivity between the two subsets of firms. The magnitude of the negative congestion effect created by information depends on the difference in the wage between the two subsets of firms. If the difference in average productivity between groups is sufficiently small (large), then the welfare in the directed search equilibrium is lower (higher) than in the random search.

Independently of the structure of information, the decentralized allocation is constrained inefficient. More specifically, if a social planner –who cannot directly observe the firm's type– were to offer a menu of queues and transfers to the firms and choose the worker's application probabilities conditional on the firm's choices, she would achieve an allocation that Pareto improves over the decentralized allocation. There are two causes for the inefficiency of the decentralized allocation. First, when a worker submits her application she creates a negative congestion externality on all the other people applying for the same position. Secondly, when a firm posts a certain message it changes the informative content of other firm's communication.

Relation to the Literature The search literature has primarily focused on two polar representations of technology and contracts in the economy: random search and wage posting. According to the random search view –best described in Mortensen (1970) – a worker has no information about the needs and wages of each firm and to determine whether her skills meet the requirements for a particular vacancy and to ascertain the wage offered to fill the opening, she must search it. In this class of models, communication is not feasible before a relationship-specific investment has occurred and therefore the equilibrium outcome coincides with the outcome of a game where workers are constrained to randomize uniformly across firms and is generally inefficient. According to the wage posting view –as exemplified in the seminal work by Montgomery (1993), Moen (1997), Burdett Shi and Wright (2001) –, firms can costlessly communicate with workers before the application stage and the content of communication is contractually binding. In this class of models, search is directed and the allocation of labor is constrained efficient.

This paper shows that in the intermediate environment where ex-ante communication is feasible but not enforceable, both random and directed search can be obtained depending on the fundamentals of the economy. When put in the context of the existing literature, the result can be interpreted as a proof by counterexample that communication is not a sufficient condition for the existence of a directed search equilibrium and that, on the other hand, lack of commitment is not a sufficient condition for random search. In turn, because the empirical evidence that firms commit to a contract when advertising a job opening is at best mixed, the result suggests that directed search need not be a more realistic description of the world than random search. Moreover, by uncoupling the assumptions of communication and commitment, the paper proves that directed search need not be welfare improving over random search because the congestion effects created by information transmission cannot be internalized through prices.

Indirectly, this paper also contributes to the recent debate on the cyclical behavior of unemployment and vacancies. In a very influential paper, Shimer (2005) argues that the random search model predicts too much volatility in the wages and too little volatility in unemployment over the business cycle. In the model, the quasi-rents are divided according to the generalized Nash bargaining solution and when a shock to aggregate productivity occurs wages adjust directly because of the change in a match's surplus and indirectly because of the change in the worker's outside option. For reasonable choices of the model's parameters, these two procyclical effects are such that wages absorb most of the productivity fluctuations.

The results of this paper suggest that the endogeneity of the information structure can amplify the unemployment effects of productivity fluctuations and dampen the wage effects. In fact, the comparative statics in the paper show that a positive shock to the tightness of the labor market –which in turn could be driven by a positive aggregate productivity shock– might lead to a reduction in the informativeness of the cheap-talk, an increase in the informational rents extracted by the firms during the bargaining stage and a reduction in the congestion effects at the application stage.

Finally, this paper relates to the industrial organization literature. In a series of influential papers, Nelson (1970, 1974) argued that inherently uninformative advertisement can still be useful to consumers if there exists an equilibrium relationship between advertisement expenditures and the quality of the good traded. In order to sustain a signaling equilibrium, firms providing high-quality goods must have a stronger return from increased sales than low-quality firms. Nelson argues that a good reason for differential returns is repeat purchases. These ideas are formalized by Milgrom and Roberts (1986) who study a (partial equilibrium) economy where a monopolistic firm has private information about the good's quality it sells and chooses advertisement expenditures and prices. They show that advertisement might be used in a separating equilibrium when the firm's quality is high.

My paper offers an alternative rationale behind the relationship between search strategies and non-contractual advertisement. Separating equilibria are sustained because firms with high productivity jobs have a stronger preference for filling a vacancy than low productivity firms. Also in my model, the cost of signaling high-quality does not take the form of advertisement expenditures, but of workers becoming more demanding in the bargaining stage. Fundamentally, the differences with respect to Milgrom and Roberts are due to the focus on the labor market rather than experience goods markets.

In spirit –if not in the framework– my paper is closest to Bagwell and Ramey (1994) who address the topic of the informative content of advertisement in a general equilibrium environment where search is time consuming. They study the joint determination of the selling technology (which affects the marginal cost of sales), the product line (which determines sales for any given number of costumers visiting the store), the price and the advertisement expenditures for the retailing firms. They find conditions under which a greater investment in selling technology, an expansion in the product line, and lower prices are mutually reinforcing responses to an increase in the expected market share. They show that in the set of SPE equilibria, there always exist (among others) both equilibria where consumers ignore the advertisement expenditures (random search) and equilibria where they visit the firm with the highest advertisement expenditures (directed search). In contrast to my paper, in their context the directed search equilibrium always exists and is always more efficient than random search. Intuitively, because there are increasing returns to scale in the retail sector, it is always possible to find an equilibrium where only one firm enters the market and any extra-profits are eliminated through advertisement expenditures.

Structure of the paper Section 2 describes the model economy and the equilibrium concept. Section 3 first derives the outcome of the bargaining game, then identifies a set of necessary and sufficient conditions for the existence of an equilibrium where N informationally distinct messages are posted, and finally derives the relationship between the fundamentals of the economy, the nature of search and the equilibrium allocation. Section 4 discusses the efficiency properties of the decentralized allocation and compares the welfare properties of random and directed search equilibria. Section 5 briefly concludes by discussing the direction of future research.

2 The Model

There are a continuum of heterogeneous firms with measure m –each with one job opening– and a continuum of homogeneous workers with measure n. For every i in $[0, m]$, firm i has private information over its type $y(i)$, which is a random variable drawn from a commonly known distribution function $F(y)$ that has positive density over the support $\left[\underline{y}, \overline{y}\right]$ with $\underline{y} > 0$.

In period $t = 0$, every firm i chooses a message from the set $S = \{s(1), s(2), ...s(K)\}$, with $K > 2$, to advertise its vacancy. In period $t = 1$, after having observed the distribution of messages, every worker j chooses a firm $i \in [0, m]$ where to search for a job. If firm i does not receive any applicants for the job, then its vacancy remains unfilled and the firm's payoff is normalized to 0. If some workers show up at firm's i location, then one of them is randomly chosen to fill the vacancy while the others are sent back to the unemployment pool and receive a zero payoff. If a match is created, firm i and worker j enter an alternating offer bargaining game à la Binmore, Rubinstein and Wolinsky (1986) in order to determine the terms of trade. More specifically, in period $t = 2$ the worker advances a wage demand w_2. In period $t = 3$, if the firm accepts the worker's demand, an action denoted by Y, then the game ends and the worker's and firm's payoffs are respectively given by w_2 and $y(i) - w_2$. On the other hand, if the firm rejects the wage demand w_2, with probability $1 - \exp(-\beta\Delta)$ the negotiation breaks down and the game ends with both parties receiving a zero payoff and with probability $\exp(-\beta\Delta)$ the firm's counteroffer w_3 reaches the worker. In period $t = 4$, the worker can accept w_3 or advance a wage demand w_4. In the former case the game ends with payoffs w_3 and $y(i) - w_3$. In the latter case, with probability $1 - \exp\left(-\left(1 - \beta\right)\Delta\right)$ the negotiation breaks down and both parties receive a zero payoff and with probability $\exp(-(1-\beta)\Delta)$ the worker's wage demand w_4 reaches the firm. The process continues without a deadline. I will consider the limit outcome of the bargaining game as the time delays $\beta\Delta$ and $(1-\beta)\Delta$ between rounds of offers converge to zero for a constant $\beta \in (0,1)$.

Denote with $H^t_{w,j} = H^{t-1}_{w,j} \cup H_{w,j,t}$ the set of histories observed by worker j up to period t and with $h^t_{w,j}$ an element of the set. By convention, $H^0_{w,j}$ is equal to the empty set. An observation $h_{w,j,1}$ is a mapping from the firms' names into messages and $H_{w,j,1}$ is the set of mappings from $[0,1]$ to S. The element $h_{w,j,2}$ is given by the identity of worker's j trading partner and $H_{w,j,2}$ is the interval $[0,m]$. For $t > 2$, $h_{w,j,t}$ is either a wage demand by worker j (when t is odd) or a wage offer from firm $h_{w,j,2}$ (when t is even) and $H_{w,j,t}$ is $\mathbb{R}_+$. The set of histories $H^t_{f,i}$ observed by firm i up to period t is defined in an analogous way. The strategy of worker j is a sequence of functions $\sigma^t_{w,j} : H^t_{w,j} \to A_{w,t}$, where $A_{w,1}$ is the set of probability distributions over the interval $[0,m]$ and $A_{w,t}$ is equal to $\Delta(\{Y\} \cup \mathbb{R}_+)$ for $t = 2, 4, ...$ The strategy of firm i is a sequence of functions $\sigma^t_{f,i} : H^t_{f,i} \to A_{f,t}$, where $A_{f,0}$ is the set of probability distributions over S and $A_{w,t}$ is equal to $\Delta(\{Y\} \cup \mathbb{R}_+)$ for $t = 3, 5, ...$ Denoting with W the set of probability distributions on $[\underline{y}, \overline{y}]$, then the beliefs of the worker j about the productivity of firm i are given by a function $g_{i,j} : \cup^{\infty}_{t=1} H^t_{w,j} \to W$. The strategies $\sigma = \{\sigma_{w,j}, \sigma_{f,i}\}$ constitute a sequential equilibrium together with the belief updating function $g = \{g_{i,j}\}$ if: (i) the strategies $\sigma_{w,j}$ and $\sigma_{y,i}$ are sequentially rational; (ii) (g, σ) are the limit of a sequence of completely mixed strategies.

Following Montgomery (1991) and Burdett, Shi and Wright (2001), I restrict attention to symmetric equilibria where neither the strategy of the worker not the strategy of the firm depend on their names j and i. The symmetry of the worker's strategy implies that workers cannot coordinate on who will visit which location and it generates equilibrium search frictions. The symmetry of the firm's strategy implies that the interpretation that workers give to the message posted by a certain firm does not depend on the firm's identity. Finally, in order to make the analysis of the bargaining game manageable, I will focus on equilibria with the property that the advertising strategy σ^0_y is convex in the firm's type (in the sense that if y_1 and y_2 post the message $s(k)$ with positive probability, then every firm with productivity $y \in (y_1, y_2)$ posts $s(k)$ with positive probability as well), the strategies σ^t_w and σ^t_y are pure for $t \geq 2$ and the updating rule g is such that no revision increases the support of the belief distribution.

Along with these three substantive restrictions, it is possible to reduce the set of equilibria by normalizing the number and ordering of the messages used in the cheap-talk game. First, notice that if two messages –let's say $s(1)$ and $s(2)$– are used in equilibrium and generate posterior beliefs such that both the worker and the firm expect the same payoff from the bargaining game, then there exists another equilibrium with the same final allocation where only $s(1)$ is used. Secondly notice that, if there exists an equilibrium where two messages $s(1)$ and $s(2)$ are used, then there is an equivalent equilibrium where $s(2)$ takes the place of $s(1)$ and viceversa. Therefore, the set of equilibrium allocations can be completely characterized by considering equilibria where the messages posted with positive probability by some firms are $\{s(k)\}_{k=1}^{N}$, for $N = 1, 2, ...K$, every posted message leads to a different (expected) bargaining outcome, and the sequence is non-decreasing in the worker's expected payoff.

3 Characterization of the Set of Equilibria

3.1 The Outcome of the Bargaining Game

In this subsection, I study the sequential equilibria of the alternating offer bargaining game between a firm i whose vacancy has productivity $y \in [\underline{y}, \overline{y}]$ and a worker j whose initial beliefs $g_{i,j}\left(h_{w,j}^{2}\right)$ are given by a continuous cumulative distribution G with support $[l, h]$, $\underline{y} \leq l \leq h \leq \overline{y}$. The main result is that, under mild stationarity and monotonicity conditions on the seller's strategy and some natural restrictions on the off-equilibrium conjectures of the worker, the outcomes of any sequence of equilibria of the bargaining game converge towards a unique outcome as the time interval between offers goes to zero. The limiting outcome is such that a type $y \geq \beta \cdot l$ trades at the wage $\beta \cdot l$ and reaches the agreement instantly, while a type $y < \beta \cdot l$ does not trade.

It is useful to begin the characterization with some (well-known) general properties that hold in any sequential equilibrium of the bargaining game and after any history h^t that is a subsequent of $h_{w,j}^{2}$. First, because the cost of delaying agreement increases with the firm's type y while the benefit is independent from y, it is immediate to verify that –in any sequential equilibrium and conditional on having reached any history h^t– the wage that a firm y agrees to pay is non-decreasing in y and the period in which the agreement takes place is non-increasing in y. Secondly, as shown in Grossman and Perry (1986b), if the history h^t generates posterior beliefs $g_{i,j}(h^t)$ with support over the interval $[b, a]$, then any type $y \in [b, a]$ will reject the demand w of a worker if

$$w > \frac{1 - \exp(-\beta\Delta)}{1 - \exp(-\Delta)} \cdot a \equiv \rho \cdot a$$

and accept it if $w < \rho \cdot b$. Similarly, in any sequential equilibrium the worker rejects a wage offer w if w is smaller than $\exp(-(1-\beta)\Delta)(\rho \cdot b)$ and accepts it if w is greater than $\exp(-(1-\beta)\Delta)(\rho \cdot a)$. Not much more can be said about the outcome of the bargaining game, because the notion of sequential equilibrium does not impose enough structure on the relationship between the history and the players' strategies and on how the worker's beliefs are updated after the firm takes an off-equilibrium action.

In order to restrict the way the firm's bargaining strategy depends upon the observed sequence of wage demands and wage offers, I follow Gul and Sonnenschein (1988). The first condition that they consider requires that σ_y^t depend on the history h^t exclusively through the effect that h^t has on the worker's beliefs $g_{j,i}(h^t)$. The second condition requires that, if after some history h^{t_1} the worker has more optimistic beliefs about the firm's type than after some other history h^{t_2}, then the firm's acceptance wage is at least as high in the former case than in the latter.

Condition 1: (Stationarity) *Let t_1 and t_2 be even natural numbers greater than 2 and let h^{t_1} and h^{t_2} be two histories such that $g_{i,j}(h^{t_1}) = g_{i,j}(h^{t_2})$. If w_{t_1} and w_{t_2} are such that $\sigma_y^{t_1+1}(\{h^{t_1}, w_{t_1}\}) = Y$ if and only if $\sigma_y^{t_2+1}(\{h^{t_2}, w_{t_2}\}) = Y$, then $\sigma_y^{t_1+1}(\{h^{t_1}, w_{t_1}\})$ is equal to $\sigma_y^{t_1+1}(\{h^{t_1}, w_{t_2}\})$ for all $y \in [\underline{y}, \overline{y}]$.*

Condition 2: (Monotonicity) *Let t_1 and t_2 be even natural numbers greater than 2 and let h^{t_1} and h^{t_2} be two histories. If $g_{i,j}(h^{t_2})$ is a conditional distribution of $g_{j,i}(h^{t_1})$ on the interval $y \in [\underline{y}, \hat{y}]$, with $\hat{y} \leq h$, then for all $w_{t_1} \in \mathbb{R}_+$ there is a $w_{t_1} \geq w_{t_2}$ such that $\sigma_y^{t_1+1}(\{h^{t_1}, w_{t_1}\}) = Y$ if and only if $\sigma_y^{t_2+1}(\{h^{t_2}, w_{t_2}\}) = Y$ for all $y \in [\underline{y}, \overline{y}]$.*

In order to restrict the worker's beliefs after an off-equilibrium action is observed, I follow the insights of Grossman and Perry (1986a and 1986b). In general, the worker can insist on a wage offer w strictly greater than $\exp(-(1-\beta)\Delta) \cdot \rho \cdot a$, even though she knows that the offer comes from $y \in [l, a]$, by "threatening" to revise her beliefs optimistically if anything but w is offered. Similarly, there are equilibria where every firm $y \in [l, a]$ accepts a wage demand w greater than $\rho \cdot l$ because if the firm were to refuse to trade the worker would not revise her beliefs downwards. In order to rule out these unreasonable equilibria of the bargaining game, I impose the following restrictions.

Condition 3: (Monotonic Conjectures) *Suppose that after history h^t, the worker's beliefs $g_{i,j}(h^t)$ have support on $[l, a]$, with $l \leq a \leq h$. If t is odd and the equilibrium prescribes* *that w_t is offered by all $y \in [l,a]$ and accepted by the worker, then w_t is smaller or equal than* $\exp\left(-\left(1-\beta\right)\Delta\right)\left(\rho\cdot a\right)$. *If t is even and the equilibrium prescribes that w_t is accepted by all $y \in [l,a]$, then the posterior belief $g_{i,j}\left(h^t, w_{t+1}\right)$ is degenerate at l.*

In any sequential equilibrium, if a firm y_1 trades at the wage $w\left(y_1\right)$, then the expected payoff from rejecting the wage demand or making an unacceptable counteroffer has to be smaller than $\left(y_1 - w\left(y_1\right)\right)$. Because in equilibrium higher types trade earlier and the off-equilibrium conjectures are monotonic, then in any sequential equilibrium that satisfies condition 3, the firm can refuse to trade at $w\left(y_1\right)$ and induce the worker to revise her beliefs downwards. More specifically, there exist a firm's action such that the worker's posterior beliefs are given by the prior $g_{j,i}\left(h^t\right)$ conditional on $y \leq \hat{y}$ for some $\hat{y} \leq y_1$. If the firm takes this action, it can reach an agreement with the worker by offering her a wage equal to $\exp(-\left(1-\beta\right)\Delta)\left(\rho\cdot\hat{y}\right)$. Using a revealed preference argument, it is possible to conclude that for every $y_1 \in [l,h]$ the wage $w\left(y_1\right)$ has to be contained in the interval

$$w(y_1) \in \left[e^{-(1-\beta)\Delta}\left(\frac{1-e^{-\beta\Delta}}{1-e^{-\Delta}}\right)\cdot l, \left(1-e^{-\Delta}+e^{-(2-\beta)\Delta}\left(\frac{1-e^{-\beta\Delta}}{1-e^{-\Delta}}\right)\right)\cdot y_1\right]. \qquad (1)$$

If a sequential equilibrium satisfies conditions 1–3, then the assumptions of theorem 1 in Gul and Sonnenschein (1988) are satisfied and I can conclude that for any $\epsilon > 0$ there is a $\overline{\Delta}\left(\epsilon\right) > 0$ such that for all $\Delta < \overline{\Delta}\left(\epsilon\right)$ the probability that the game terminates within ϵ time is at least $\left(1-\epsilon\right)$. Therefore, for $\Delta \to 0$, almost every firm $y \in [l,h]$ trades immediately and for approximately the same wage, which –in light of (1)– has to be arbitrarily close to $\beta \cdot l$. In order to formalize this argument, let the set of sequential equilibria that satisfy conditions 1–3 be denoted by $\Theta\left(\Delta; G, \beta\right)$, and let θ represent an element of the set. Denote with $\Pi\left(\theta, y; G, \beta\right) = \left(w, \tau\right)$ the wage outcome and the time it takes for an agreement to be eventually reached, given that the worker and the firm are playing according to θ and the firm has productivity y.

Proposition 1: (Coase Conjecture) *Fix a worker's bargaining power β in $(0,1)$ and a prior belief G. Consider a sequence $\{\Delta(n)\}_{n=1}^{\infty}$, such that* $\lim_{n\to\infty}\Delta(n) = 0$, *and let $\theta(n)$ belong to $\Theta(\Delta(n); G, \beta)$ for all n. The outcome sequence $\{\Pi\left(\theta\left(n\right), y; G, \beta\right)\}_{n=1}^{\infty}$ converges and its limit is*

$$\Pi\left(y; G, \beta\right) = \begin{cases} \left(\beta\cdot l, 0\right) & \text{if } y \geq \beta\cdot l, \\ \left(0, \infty\right) & \text{otherwise.} \end{cases} \qquad (2)$$

Proof: In the appendix. Q.E.D.

For all $y \in [l,h]$, as the time interval between two offers becomes arbitrarily small, the outcome of the asymmetric information bargaining game converges to the outcome of the full information version of the game played between the worker and a firm with the lowest productivity on the support of worker's beliefs. When –off the equilibrium path– a firm with productivity $y \notin [l, h]$ enters the negotiation, the outcome is immediate agreement at the wage $\beta \cdot l$ if $y \geq \beta \cdot l$ and no trade if $y < \beta \cdot l$.

3.2 Necessary and Sufficient Conditions for a N-message Equilibrium

As discussed in section 2, it is without loss of generality that we can restrict attention to equilibria where: (i) the messages posted with positive probability are $\{s(k)\}_{k=1}^{N}$, for some N between 1 and K; (ii) every posted message leads to a different (expected) bargaining outcome; (iii) the sequence $\{s(k)\}_{k=1}^{N}$ is non-decreasing in the worker's expected payoff. In light of proposition 1, condition (ii) is satisfied if and only if the worker's posterior beliefs about a firm that have posted the message $s(k)$ has a lower bound $\underline{y}(k)$ which is different for every $k = 1, 2, ...N$. Also, proposition 1 implies that condition (iii) is satisfied if and only if the sequence $\left\{\underline{y}(k)\right\}_{k=1}^{N}$ is strictly increasing. This subsection derives a set of necessary and sufficient conditions for the existence of a general equilibrium where N informationally different –in the sense of condition (ii)– messages are used.

3.2.1 Necessary Conditions

Suppose that there exists an equilibrium where N informationally different messages are posted with positive probability. For $k = 1, 2, ...N$, denote with $G^*(y|k)$ the worker's beliefs about a firm that posts the message $s(k)$, with $\underline{y}^*(k)$ the lower bound on $G^*(y|k)$ and with $w^*(k) = \beta \cdot \underline{y}^*(k)$ the worker's expected wage.

In general, worker's j application strategy $\sigma_{w,j}^{1}$ is a map from the history $h_{w,j}^{1}$ to the set of distribution function over the interval $[0, m]$. When this strategy is restricted to be independent from the worker's identity, then the worker's optimal application is such that she visits with the same probability every firm that posts the same message. In particular, along the equilibrium path, $a_{j,w}^{1}$ can be represented by a vector $\{q^*(1), q^*(2), \ldots q^*(N)\}$, where $q^*(k) \in [0, 1]$ denotes the probability that the worker applies to a firm that posts the message $s(k)$ and $\sum_{k=1}^{N} q^*(k) = 1$.

Given the application strategy $\{q^*(k)\}_{k=1}^{N}$ and the measure $M^*(k)$ of firms that post $s(k)$, the number of workers that visit each vacancy $s(k)$ is a Poisson random variable with mean

$$\lambda^*(k) = (M^*(k))^{-1} (n \cdot q^*(k)) . \tag{3}$$

From the perspective of a worker, when she chooses to visit a vacancy advertised with $s(k)$, she expects to receive the job with probability $\lambda^*(k)^{-1} \cdot (1 - \exp(-\lambda^*(k)))$ and to obtain the wage $w^*(k)$. Therefore, the equilibrium application $\{q^*(k)\}_{k=1}^N$ has to satisfy the following optimality condition

$$\frac{1 - \exp(-\lambda^*(k))}{\lambda^*(k)} w^*(k) = \max_{l=1,2,...N} \frac{1 - \exp(-\lambda^*(l))}{\lambda^*(l)} w^*(l), \quad \text{if } \lambda^*(k) > 0. \tag{4}$$

In a N-message equilibrium, the advertisement strategy σ_y^0 can be represented as a vector $\{p^*(k|y)\}_{k=1}^N$, where $p^*(k|y) \in [0,1]$ denotes the probability that firm y posts $s(k)$ and $\sum_{k=1}^N p^*(k|y) = 1$. When firm y posts the message $s(k)$, it receives at least one applicant with probability $P^*(k)$ equal to $1 - \exp(-\lambda^*(k))$ and it obtains the payoff $\max\{y - w^*(i), 0\}$ from the bargaining game. Therefore, for every $y \in [\underline{y}, \overline{y}]$ the equilibrium advertisement strategy $\{p^*(k|y)\}_{k=1}^N$ has to satisfy the following optimality condition

$$P^*(k)(\max\{y - w^*(k), 0\}) \geq \max_{l=1,2,...N} P^*(l)(\max\{y - w^*(l), 0\}), \quad \text{if } p^*(k|y) > 0. \tag{5}$$

In any N-message equilibrium, the worker's posterior belief $G^*(y|k)$ about the productivity of a vacancy that is advertised with the message $s(k)$ must be equal to the conditional distribution of $F(y)$ given that firms follow the posting strategy $\{p^*(k|y)\}_{k=1}^N$. More formally, the following condition holds for $k = 1, 2, ...N$

$$\begin{aligned} G^*(y|k) &= (M^*(k))^{-1} \left(\int_{\underline{y}}^{y} p^*(k|y_s) dF(y_s) \right), \\ M^*(k) &= \int_{\underline{y}}^{\overline{y}} p^*(k|y) dF(y). \end{aligned} \tag{6}$$

Moreover, it follows from (6) and the assumption of convex advertisement strategies, that the lower bound $\underline{y}^*(k)$ of the posterior belief $G^*(y|k)$ is equal to the infimum y over $[\underline{y}, \overline{y}]$ such that $p^*(k|y)$ is greater than zero.

The optimality condition (4) on the worker's application strategy, together with the assumption that $\{w^*(k)\}_{k=1}^N$ is a strictly increasing sequence, implies that the average queue length at a firm posting $s(l)$ is smaller or equal than at a firm posting $s(k)$, for all $N \geq k > l$. For a moment, suppose that the message $s(k)$ is the lowest that generates a positive queue length $\lambda^*(k)$ and that $k > 1$. Then, the optimality condition (5) on the firm's advertising strategy implies that every firm y with productivity greater than $\beta \underline{y}^*(k)$ must be making strictly positive expected profits in equilibrium. In particular, this is true for the firms y in the interval $\left(\beta\underline{y}^*(k), \underline{y}^*(k)\right)$, which do not post any of the messages $\{s(k), ...s(N)\}$. Therefore, there exists a message $s(l)$, with l smaller than k, such that $P^*(l) > 0$. The contradiction that we have just reached leads to the following lemma.

LEMMA 1: (Positive Profits) *In any N-message equilibrium:*
(i) If $p^*(k|y) > 0$, then $P^*(k)(y - w^*(k)) > 0$;
(ii) $P^*(k)$ and $\lambda^*(k)$ are strictly positive for $k = 1, 2, \ldots N$.

PROOF: In the appendix. Q.E.D.

The first implication of lemma 1 is that, because every message posted in equilibrium attracts some applicants, the worker must obtain the same expected utility if she applies to a vacancy advertised with $s(k)$ as if she applies to a vacancy advertised with $s(l)$. More formally, in any N-message equilibrium and for all l and k between 1 and N, the following condition has to hold

$$\frac{1 - \exp(-\lambda^*(k))}{\lambda^*(k)} w^*(k) = \frac{1 - \exp(-\lambda^*(l))}{\lambda^*(l)} w^*(l). \tag{7}$$

The second implication of lemma 1 is that, because every firm makes a positive profit in equilibrium, the optimality condition (5) is satisfied if and only if the message $s(k)$ maximizes the payoff function $P^*(l) \cdot (y - w(l))$ for $1 \leq l \leq N$. According to such payoff function, the gain from posting a message $s(k+1)$ rather than $s(k)$ is strictly increasing in y, while the cost is independent from y. Therefore, the single crossing property holds: if y_1 is indifferent between $s(k)$ and $s(l)$ and $k > l$, then all types higher than y_1 strictly prefer $s(k)$ and all types lower than y_1 strictly prefer $s(l)$. The single crossing property is used in lemma 2 in order to prove that the message $s(k)$ is posted exclusively by firms in the interval $\left[\underline{y}^*(k), \underline{y}^*(k+1)\right]$.

LEMMA 2: (Cutoff Property) *In any N-message equilibrium and for all $1 \leq k \leq N-1$, $p^*(k|y) = 1$ if y belongs to the interval $\left(\underline{y}^*(k), \underline{y}^*(k+1)\right)$ and $\underline{y}^*(k+1)$ is indifferent between $s(k)$ and $s(k+1)$.*

PROOF: In the appendix. Q.E.D..

The analysis of the necessary conditions of a N-message equilibrium is summarized by the following proposition.

PROPOSITION 2: (Necessary Conditions) *In any N-message equilibrium, the following $2N$*

equations and N inequalities are satisfied:

$$\beta \underline{y}^*(k)\frac{1-e^{-\lambda^*(k)}}{\lambda^*(k)} = \beta \underline{y}^*(l)\frac{1-e^{-\lambda^*(l)}}{\lambda^*(l)} \equiv U^*, \qquad \text{all } k, l,$$

$$\left(1-e^{-\lambda^*(k)}\right)\left(\underline{y}^*(k+1)-\beta \underline{y}^*(k)\right) = (1-e^{-\lambda^*(k+1)})\underline{y}^*(k+1)(1-\beta), \quad k=1,...N-1,$$

$$\underline{y} = \underline{y}^*(1) < ... < \underline{y}^*(N) \leq \overline{y},$$

$$b = \sum_{i=1}^{N} \lambda^*(k)\left[F\left(\underline{y}^*(k+1)\right) - F\left(\underline{y}^*(k)\right)\right] . \qquad \text{(N\&S)}$$

3.2.2 Sufficient Conditions

Next, I suppose that $\{\underline{y}^*(k);\lambda^*(k)\}_{k=1}^{N}$ is a solution to the system of $2N$ equations and N inequalities in (N&S) and I construct a N-message sequential equilibrium.

In the putative equilibrium, the advertising strategy $\{p^*(k|y)\}_{k=1}^{K}$ of firm y is to post the message $s(k)$ with probability one if its productivity lies in the non-empty interval $\left(\underline{y}^*(k), \underline{y}^*(k+1)\right)$. Consistently with the putative advertising strategy, I assume that worker's posterior belief $G^*(y|k)$ about a firm that has posted the message $s(k)$ is given by the distribution $F(y)$ conditional on y being greater than $\underline{y}^*(k)$ and smaller than $\underline{y}^*(k+1)$, for all $k=1,2,...N$. Off the equilibrium path, when a worker observes a message $s(k)$, with $k>N$, her posterior belief $G^*(y|k)$ is equal to the prior $F(y)$. Consistently with proposition 1 and the worker's posterior beliefs, I assume that the outcome of the bargaining game between the worker and a firm with productivity y are $\beta \underline{y}^*(k)$ and $\max\left\{y-\beta \underline{y}^*(k), 0\right\}$, if the firm has posted a message $s(k)$ with $k \leq N$. When a worker bargains with a firm that has posted an off-equilibrium message, the expected payoffs for the two parties are assumed to be $\beta \underline{y}$ and $\left(y-\beta \underline{y}\right)$. Conditional on almost every firm following the putative advertising strategy, I conjecture that the worker's application strategy is such that she visits a vacancy advertised with either $s(1)$ or one of the messages $\{s(k)\}_{k=N+1}^{K}$ with probability $\left(b^{-1}\lambda^*(1)\right)\cdot\left[F\left(\underline{y}^*(2)\right)-F\left(\underline{y}^*(1)\right)\right]$ and that she visits a vacancy advertised with $s(k)$, with k between 2 and N, with probability $\left(b^{-1}\lambda^*(k)\right)\cdot\left[F\left(\underline{y}^*(k)\right)-F\left(\underline{y}^*(k)\right)\right]$.

Given the putative application and advertisement strategies, the average queue length at a firm that posts $s(k)$ is given by $\lambda^*(k)$ for $k=1,...N$ and by $\lambda^*(1)$ for $k=N+1,...K$. Therefore, a worker that applies to a vacancy advertised with $s(k)$ obtains the job with probability $(\lambda^*(k))^{-1}\cdot(1-\exp(-\lambda^*(k)))$ and receives the wage $w^*(k)$ for $k=1,...N$.

When a worker applies to a $s(k)$ vacancy with $k > N$, her expected utility is the same that she would obtain from applying to a $s(1)$ vacancy. Because $\{\underline{y}^*(k); \lambda^*(k)\}_{k=1}^N$ solves equations (N&S.4), the putative application strategy is feasible. Because $\{\underline{y}^*(k); \lambda^*(k)\}_{k=1}^N$ solves equations (N&S.1), every vacancy delivers the same expected utility and the putative application strategy is optimal.

Given the putative application strategy and the worker's posterior beliefs, the firm's advertisement strategy is optimal if for all y in the interval $\left(\underline{y}^*(k), \underline{y}^*(k+1)\right)$ and for all $l = 1, 2...N$

$$D(s(k), s(l), y) \equiv (1 - e^{-\lambda^*(k)})(y - w^*(k)) - (1 - e^{-\lambda^*(l)})(y - w^*(l)) \geq 0. \tag{8}$$

The expected profit differential $D(s(N), s(N-1), y)$ is strictly increasing in the firm's type y and, according to equation (N&S.2), it is equal to zero for $y = \underline{y}^*(N)$. This implies that a type y smaller than $\underline{y}^*(N)$ never posts $s(N)$ and a firm y greater than $\underline{y}^*(N)$ never posts $s(N-1)$. Similarly, the expected profit differential between posting $s(N-1)$ and $s(N-2)$ is strictly increasing in y and it is equal to zero for $y = \underline{y}^*(N-1)$. Therefore, a type y smaller than $\underline{y}^*(N-1)$ –which in turn is smaller than $\underline{y}^*(N)$– does not post either $s(N-1)$ nor $s(N)$ and a firm y greater than $\underline{y}^*(N-1)$ never posts $s(N-2)$. Iterating this argument, we can conclude that the putative advertisement strategy $\{p^*(k|y)\}_{k=1}^K$ is optimal.

Proposition 3: (Sufficient Conditions) *If* $\left\{\underline{y}^*(k); \lambda^*(k)\right\}_{i=1}^N$ *is a solution to (N&S), then there exists a N-message equilibrium.*

3.3 Strategic Information Transmission in Partial Equilibrium

In this subsection, I treat the value of search U parametrically and find the solutions to the system of equations (N&S.1)-(N&S.3) for $N = 1, 2, ...$ This approach affords an interesting economic interpretation, because whenever U belongs to the interval $\left(0, \beta\underline{y}\right)$ –a condition which guarantees that profits are positive for every firm's type– the solutions to $(N\&S.1)$-$(N\&S.3)$ identify the equilibria of the cheap-talk and bargaining game played between a single firm and a workforce whose search behavior is represented by the labor supply curve

$$w = \frac{\lambda}{(1 - \exp(-\lambda))} U. \tag{9}$$

Equation (9) defines the wage $w(\lambda; U)$ that a firm has to promise to the applicants, in order to attract on average λ, $\lambda > 0$, when the value of search is U. Also, equation (9) can be used to define an inverse labor supply curve $\lambda(w; U)$ that returns the average length of the applicants' queue at a firm that promises the wage $w \in \left[\beta\underline{y}, \infty\right)$.

In any equilibrium of the cheap-talk game, the firm has the option to post the message $s(1)$, attract an average of $\lambda(1) = \lambda\left(\beta\underline{y}; U\right)$ applicants and pay them a wage $w(1) = \beta\underline{y}$. Suppose that there exists a type $\hat{y}$ that would be indifferent between posting $s(1)$ and credibly revealing its own type, thereby attracting $\lambda\left(\beta\hat{y}; U\right)$ applicants and conceding on the wage $\beta\hat{y}$. If such $\hat{y}$ exists, then –because of the single crossing property– all types in the interval $\left[\underline{y}, \hat{y}\right)$ strictly prefer $(w(1), \lambda(1))$ over $(\beta\hat{y}, \lambda(\beta\hat{y}; U))$ and all types higher than $\hat{y}$ would rather pool with $\hat{y}$ than post $s(1)$. Therefore, if there is a $\hat{y}$ indifferent between $(w(1), \lambda(1))$ and $(\beta\hat{y}, \lambda(\beta\hat{y}; U))$, there exists an equilibrium of the cheap-talk game with two messages. Conversely, if there exists an equilibrium of the cheap-talk game with two messages, then the type that is indifferent between posting $s(2)$ and $s(1)$ has to be $\beta^{-1}w(2)$, or else the worker's beliefs would not be consistent with the outcome of the bargaining game.

Iterating the argument, we can conclude that an equilibrium of the cheap-talk game with N informationally distinct messages exists if and only if there is a strictly increasing sequence of types $\left\{\underline{y}(k)\right\}_{k=1}^{N}$ where $\underline{y}(k)$ is indifferent between posting $s(k-1)$ and credibly revealing its own type, $\underline{y}(1) = \underline{y}$ and $y(N) \leq \overline{y}$. If these conditions are re-expressed in terms of queue lengths, we can conclude that an equilibrium of the cheap-talk game with N messages exists if and only if there is a monotonically increasing sequence $\{\lambda(i)\}_{i=1}^{N}$ such that $\lambda(1)$ is equal to $\lambda\left(\beta\underline{y}; U\right)$ and $\lambda(N) \leq \lambda\left(\beta\overline{y}; U\right)$ and $\Psi\left(\lambda\left(k+1\right), \lambda(k); U\right)$ is equal to zero for $k = 1, 2, ..N-1$, where the function $\Psi\left(\lambda, \lambda(k); U\right)$ is defined as the payoff differential between the queues λ and $\lambda(k)$ for a firm with productivity $y = \beta^{-1}w(\lambda; U)$

$$\Psi\left(\lambda, \lambda(k); U\right) =$$

$$= \left(1 - e^{-\lambda}\right)\left(\frac{w(\lambda;U)}{\beta} - w(\lambda; U)\right) - \left(1 - e^{-\lambda(k)}\right)\left(\frac{w(\lambda;U)}{\beta} - w(k)\right) = \tag{10}$$

$$= U\left[\lambda\left(\frac{1-\beta}{\beta}\right) - \left(\frac{\left(1-e^{-\lambda(k)}\right)}{U}\frac{w(\lambda;U)}{\beta} - \lambda(k)\right)\right].$$

The qualitative properties of the function (10) and of its zeroes are derived in lemma 3.

LEMMA 3: *Denote with $Z^{*}(x)$ the set of z such that $\Psi\left(z, x; U\right) = 0$ and $z > x$, then*
(i) $Z^{*}(x)$ is independent from U;
(ii) for x in the non-empty interval $X = \left(\log(1/\beta), h^{-1}(\beta)\right)$, $Z^{*}(x)$ is a singleton $z^{*}(x)$ and $\Psi\left(z, x; U\right)$ is greater (smaller) than zero for $z < (>) z^{*}(x)$;
(iii) for $x \leq \log(1/\beta)$, $\Psi\left(z, x; U\right)$ is greater than zero for every $z > x$ and $Z^{*}(x)$ is empty;
(iv) for $x \geq h^{-1}(\beta)$, $\Psi\left(z, x; U\right)$ is smaller than zero for every $z > x$ and $Z^{*}(x)$ is empty;
(v) the function $z^{*}(x)$ is continuous and strictly decreasing over X and at the boundaries $\lim_{x\to\log(1/\beta)} z^{*}(x) = \infty$ and $\lim_{x\to h^{-1}(\beta)} z^{*}(x) = h^{-1}(\beta)$.

PROOF: In the appendix. Q.E.D.

The results in lemma 3 have a clear economic interpretation. Suppose that if the firm posts the message $s(k)$, then workers expect the wage $w(k)$ and on average $\lambda(k)$ applicants show up for the job. Let's consider the incentives for and the credibility of communication for the types in the interval $\left(\beta^{-1}w(k), \infty\right)$. When $\lambda(k)$ is smaller than $\log(1/\beta)$, the probability of filling a vacancy by posting $s(k)$ is so small that every type $y > \beta^{-1}w(k)$ would strictly prefer $(\beta y, \lambda(\beta y; U))$ over $(w(k), \lambda(k))$. In this case, if $\hat{y}$ was to announce its true type and promise the wage $\beta\hat{y}$, then an interval of less productive firms would pool with $\hat{y}$ and render the wage promise not credible. Therefore, for $\lambda(k) \leq \log(1/\beta)$ the types on the interval $\left(\beta^{-1}w(k), \infty\right)$ cannot separate.

When $\lambda(k)$ is greater than $h^{-1}(\beta)$, the probability of receiving an applicant by posting $s(k)$ is so high that every type $y > \beta^{-1}w(k)$ would prefer pooling with those firms posting $s(k)$ than disclosing its own type. In this case, if a wage $\hat{w} > w(k)$ was advertised by the firms that prefer $\hat{w}$ to $w(k)$, then the worker would infer that the lowest type posting $\hat{w}$ is $\hat{y} > \beta^{-1}\hat{w}$ and she would negotiate a wage greater than $\hat{w}$. Therefore, for $\lambda(k) \geq h^{-1}(\beta)$ the types in the interval $\left(\beta^{-1}w(k), \infty\right)$ cannot separate.

When $\lambda(k)$ takes a value between $\log(1/\beta)$ and $h^{-1}(\beta)$, then there is a unique type $\hat{y}$ equal to $\beta^{-1}w(z^*(\lambda(k)); U)$ that is indifferent between $(\beta\hat{y}, \lambda(\beta\hat{y}; U))$ and $(w(k), \lambda(k))$. In this case there is an equilibrium of the cheap-talk where the message $s(k+1)$ is posted by types higher than $\hat{y}$, it attracts $\lambda(k+1) = z^*(\lambda(k))$ applicants and the wage outcome at the bargaining stage is $w(k+1) = \beta\hat{y}$. The function $z^*(x)$ is illustrated in Figure 1.

Notice that the matching process between applicants and firms is such that the labor supply curve $w(\lambda; U)$ is convex in λ. In turn, this implies that the function (10) is strictly concave. Therefore, if there exists a $\hat{y} > \underline{y}$ that is indifferent between posting $s(1)$ and revealing its own type, then lower types strictly prefer $(\beta\hat{y}, \lambda(\beta\hat{y}; U))$ over $(w(1), \lambda(1))$ and higher types strictly prefer $(w(1), \lambda(1))$ over $(\beta\hat{y}, \lambda(\beta\hat{y}; U))$. When this property of firm's preferences is combined with the strict monotonicity of profits with respect to productivity, then the following ordering holds for all $y \in (\hat{y}, \infty)$

$$(\beta y, \lambda(\beta y; U)) \prec_y (w(1), \lambda(1)) \prec_y (\beta\hat{y}, \lambda(\beta\hat{y}; U)). \tag{11}$$

Because the last term in (11) is equal to $(w(2), \lambda(2))$ –in any equilibrium where the $s(2)$ message is posted–, the preference ordering in (11) implies that there is no firm with productivity $y > \hat{y}$ that is indifferent between posting the message $s(2)$ and revealing its own type. Therefore an equilibrium with more than 2 messages cannot exist.

The analysis of the cheap-talk game between a firm and a workforce that behaves according to (9) is summarized in the following proposition.

Proposition 4: (Strategic Information Transmission in Partial Equilibrium) *Take any U in $\left(0, \beta \underline{y}\right)$, then: (i) there is an equilibrium of the cheap-talk game where all types post the same message; (ii) if and only if U is such that $\lambda(1) = \lambda\left(\beta \underline{y}; U\right)$ belongs to X and $z^*(\lambda(1))$ is smaller than $\lambda(\beta \overline{y}; U)$, there is one equilibrium of the cheap-talk game where two messages are posted; (iii) there are no equilibria where more than two messages are posted.*

3.4 Information Transmission in General Equilibrium: Directed and Random Search

In the previous subsection, I have studied the effect that the value of search U has on the amount of information transmission that can be sustained in the partial equilibrium chap-talk game. In this subsection, I study the feed-back effect that information has on the value of search U. Then, by considering the two effects at once, I provide a characterization of information transmission in general equilibrium. Formally, this is equivalent to derive the solutions to ($N\&S$.1)-($N\&S$.4) and intersect them with the solutions to ($N\&S$.1)-($N\&S$.3).

Suppose that, before the application stage, the workers learn the identity of all those firms whose productivity is above some threshold $\underline{y}(2)$. Given this piece of information, the feasibility and the optimality conditions for the worker's application strategy imply that the lengths of the average queue at a bad and at a good firm –respectively denoted by $\lambda(1)$ and $\lambda(2)$– must satisfy the equations

$$\begin{aligned} &\tfrac{1-e^{-\lambda(1)}}{\lambda(1)} \beta \underline{y} = \tfrac{1-e^{-\lambda(2)}}{\lambda(2)} \beta \underline{y}\,(2) && \text{if } \lambda(1) > 0, \\ &b = \lambda(1) + (\lambda(2) - \lambda(1)) \left(1 - F\left(\underline{y}(2)\right)\right). \end{aligned} \tag{12}$$

On the one hand, the worker's indifference condition (12.1) requires that the length of the queue at a high-productivity firm has to increase with the length of the queue $\lambda(1)$ at a low-productivity firm. On the other hand, the feasibility constraint (12.2) requires that as workers apply more frequently to low productivity firms the average queue at high productivity firms falls. Therefore, for every threshold $\underline{y}(2)$ in the interval $\left[\underline{y}, \overline{y}\right]$, the system (12) has a unique solution for $(\lambda(1), \lambda(2))$.

The function $\lambda(1)$ of $\underline{y}(2)$ implied by the solution of (12) can be rightfully interpreted as the effect of information on the value of search, because U is a strictly decreasing function of $\lambda(1)$. This function is represented in Figure 2 under the label (VI). As $\underline{y}(2)$ converges to the lower bound $\underline{y}$, information becomes less and less valuable because the productivity differential between good and bad firms vanishes. At the other end of the spectrum, as $\underline{y}(2)$ converges to $\overline{y}$, information becomes worthless because it identifies a vanishing measure of good firms. When $\underline{y}(2)$ lies in the interior of the type space $[\underline{y}, \overline{y}]$, then $\lambda(1)$ is strictly smaller than b and information has positive value.

The effect that U –or equivalently $\lambda(1)-$ has on the amount of information communicated through the cheap-talk has been derived in the previous subsection. First, proposition 4 states that for every $\lambda(1) > 0$ there exists a pooling equilibrium where every type posts the message $s(1)$. With a slight abuse of notation, this equilibrium can be represented in the $\{\underline{y}(2), \lambda(1)\}$ space as the point $(\underline{y}, \lambda(1))$. Secondly, proposition 4 states that if and only if $\lambda(1)$ belongs to the interval $X = (\log(1/\beta), h^{-1}(\beta))$ and $\underline{y}(2)$ is smaller than $\overline{y}$, there is a separating equilibrium where types lower than $\underline{y}(2)$ post the message $s(1)$, higher types post $s(2)$ and the cutoff $\underline{y}(2)$ is given by

$$\underline{y}(2) = \underline{y}\frac{z^*(\lambda(1))}{\lambda(1)}\frac{(1-\exp(-\lambda(1)))}{(1-\exp(-z^*(\lambda(1))))}. \tag{13}$$

Making use of the properties of $z^*(x)$, we can conclude that: (i) the function $\underline{y}(2)$ of $\lambda(1)$ implied by equation (13) is strictly decreasing over the domain X; (ii) as $\lambda(1)$ converges to $\log(1/\beta)$, $\underline{y}(2)$ diverges towards $+\infty$; (iii) as $\lambda(1)$ converges to $h^{-1}(\beta)$, $\underline{y}(2)$ converges to $\underline{y}$; (iv) there is a $\underline{\Gamma}$ such that $\underline{y}(2)$ is equal to $\overline{y}$ if and only if $\lambda(1) = \underline{\Gamma}$. The correspondence between $\lambda(1)$ and the cutoff type $\underline{y}(2)$ induced by the equilibria of the cheap-talk game is represented in Figure 2 under the label (SIT).

The set of general equilibria of the economy is identified by the common points of (VI) and (SIT). When the ratio b of workers to vacancies is smaller than $\underline{\Gamma}$, then the value of search is too high to sustain a separating equilibrium of the cheap-talk game for any amount of information that workers might have at the application stage. Therefore, in this case, the only equilibrium of the economy features random search: the messages posted by the firms are uninformative and workers correctly ignore them and apply with the same probability to every vacancy. When b lies in the interval $[\underline{\Gamma}, h^{-1}(\beta)]$, the value of searching without any information is such that a separating equilibrium of the cheap-talk game can be sustained. Therefore, by the intermediate value theorem, there exists at least one equilibrium featuring directed search: low productivity firms and high productivity firms post different messages and workers apply with different probabilities to different vacancies. When b falls between

$h^{-1}(\beta)$ and $\overline{\Gamma}$ –a variable that is defined in theorem 1 below– the value of searching without information is too low to sustain strategic information transmission. Nevertheless, in this region directed search equilibria exist because of the positive feed-back effect of information on the value of search. In other words, the only reason why a marginal firm wants to transmit information to the workforce is because other firms are doing so as well.

Finally, when b is greater than $\overline{\Gamma}$, then the value of search is too low to sustain a separating equilibrium of the cheap-talk no matter how much information workers have. In this region, the unique general equilibrium displays random search. These results are formally proved in theorem 1.

Theorem 1: (Directed and Random Search) *There exists a unique* $\overline{\Gamma} \in \left[h^{-1}(\beta), z^*(\underline{\Gamma})\right]$ *such that (i) for all* b *there exist a* 1*-message equilibrium, (ii) if and only if* $b \in \left[\underline{\Gamma}, \overline{\Gamma}\right]$ *there exists (at least) one 2-message equilibrium.*

Proof: In the appendix. Q.E.D.

This paper is based on the observation that, in a bilateral trade economy where firms can announce the intensity of their labor demand before workers decide which location to search and the content of these announcements is neither contractually binding nor verifiable, the amount of information available to workers when looking for jobs is determined endogenously by the incentives of different firms to separate and send different messages. Theorem 1 represents the main finding of the paper: when the matching of workers and vacancies follows the urn-ball process and the rents are divided trough a "fast" alternating-offers bargaining game, the maximal informativeness of communication is a hump-shaped function of labor market tightness. This result generalizes to any reduced-form matching process where the probability of filling a vacancy as a function of the average queue length is such that the labor supply curve is convex.

3.5 The Effect of Information on Congestion, Productivity and Wages

In a directed search equilibrium, the cheap-talk reveals to the workers the location of those firms whose productivity is greater than some threshold $\underline{y}^*(2) \in \left(\underline{y}, \overline{y}\right)$. At the bargaining stage, workers use the information to obtain $\beta\left(\underline{y}^*(2) - \underline{y}\right)$ extra rents from the high productivity firms. At the applications stage, workers use the information to direct their search efforts towards higher productivity firms. The equilibrium allocation of the economy is such that a fraction $(1 - \exp(-\lambda^*(1)))/\lambda^*(1)$ of the $m\lambda^*(1)F(\underline{y}^*(2))$ workers who apply to low productivity firms is hired and receives a wage $\beta\underline{y}$, while a fraction $(1-\exp(-\lambda^*(2)))/\lambda^*(2)$ of the $m\lambda^*(2)(1 - F(\underline{y}^*(2)))$ workers applying to high productivity firms gets the job at the wage $\beta \underline{y}^*(2)$. It follows that the average job finding probability, labor productivity and wage are respectively

$$AJF_d = b^{-1}\left[\left(1-e^{-\lambda^*(1)}\right)F\left(\underline{y}^*(2)\right)-(1-e^{-\lambda^*(2)})\left(1-F\left(\underline{y}^*(2)\right)\right)\right],$$

$$ALP_d = (AJF_d \cdot b)^{-1}\left[\left(\int_{\underline{y}}^{\underline{y}^*(2)} y_s dF(y_s)\right)\left(1-e^{-\lambda^*(1)}\right)+\left(\int_{\underline{y}^*(2)}^{\overline{y}} y_s dF(y_s)\right)(1-e^{-\lambda^*(2)})\right],$$

$$AW_d = (AJF_d \cdot b)^{-1}\beta\left[\underline{y}\left(1-e^{-\lambda^*(1)}\right)F\left(\underline{y}^*(2)\right)+\underline{y}^*(2)(1-e^{-\lambda^*(2)})\left(1-F\left(\underline{y}^*(2)\right)\right)\right]. \tag{14}$$

In a random search equilibrium, all firms post the same message and workers apply with identical probability to all of them. The average job finding probability AJF_r is $b^{-1}(1-\exp(-b))$, which is greater than in a directed search equilibrium because the probability of filling a vacancy is a strictly concave function of the queue length. The average productivity of labor ALP_r is $\int_{\underline{y}}^{\overline{y}}(y_s)\,dF(y_s)$, which is obviously smaller than in a directed search equilibrium. Finally, the average wage AW_r is $\beta\underline{y}$, which is smaller than AW_d because firms can extract more informational rents at the bargaining stage when the cheap-talk is less informative. These results are formalized in the following proposition.

Proposition 5: (Information and Allocations) *For any b such that both a random and a directed search equilibrium exists, then: (i) $AJF_d < AJF_r$; (ii) $ALP_d > ALP_r$; (iii) $AW_d > AW_r$.*

Proof: In the appendix. Q.E.D.

Theorem 1 and proposition 5 imply that the comparative statics generated by a search model with endogenous information might be qualitatively very different than in a search model where the information structure is taken as given. In the standard search model of random search and bargaining à la Mortensen and Pissarides (1999), an increase in the tightness of the labor market brings about an increase in the worker's outside option –via a shortening of the average length of an unemployment spell– and, in turn, an increase in the average wage and a fall in firm's quasi rents. In the standard competitive search model à la Moen (1997), an increase in the tightness of the labor market makes firms compete harder for applicants and leads to higher posted wages and lower quasi-rents.

In the model of communication and asymmetric information bargaining studied in this paper, an increase in the tightness of the labor market affects not only the workers' probability of finding a job, but also the amount of information that workers have while searching for jobs and bargaining for a wage. As noticed in theorem 1, the sign of the change in the informativeness of communication can be either positive or negative depending on the specific choice of the points of comparison. The most interesting case occurs when a tightening of the labor market pushes the economy from directed search to a region where no information transmission can be sustained in equilibrium. Then, as proved in proposition 5, the average wage falls, the firms' quasi-rents increase and the value of search increases less –in fact it might even decrease– than in either the static random search model or the static competitive search model. Also, the fall in information leads to lower congestion effects in a way that is observationally equivalent to a shift of the matching function in a model with exogenous information. Therefore, the reduction in the unemployment rate is greater than in either the static random search or the static competitive search model.

These observations suggest that endogenous information can provide an important mechanism of amplification of unemployment fluctuations and dampening of wage volatility that –in light of the findings by Shimer (2005)– is a necessary ingredient in order to rationalize the cyclical behavior of the U.S. labor market.

4 Efficiency

Is any of the decentralized allocations derived in the previous section constrained efficient? Is there a welfare ordering between directed and random search equilibria? In order to answer these two questions, this section characterizes the set of ex-ante Pareto efficient allocations of the model economy, subject to the constraint that the assignment of workers to firms has to be independent of the worker's identity and that the productivity of the firm cannot be not directly observed.

An allocation ω is a triple $\{q(y), t_f(y), t_w\}$. The first component of ω is a function $q : [\underline{y}, \overline{y}] \to [0,1]$ which denotes the probability that a worker applies to a firm with productivity y. The second component is a function $t_f : [\underline{y}, \overline{y}] \to \mathbb{R}$ which denotes the transfer received by a firm conditional on its type y. The third component t_w denotes the transfer received by a worker. The allocation ω is said to be *feasible* if and only if $q(y)$ is a probability measure, every firm has an incentives to truthfully reveal its type and the total sum of transfers is zero or, equivalently, if and only if

$$\int_{\underline{y}}^{\overline{y}} q(y)dy = 1, \tag{PM}$$

$$\left(1 - e^{-\lambda(y)}\right) y + t_f(y) \geq \max_{\hat{y} \in \left[\underline{y}, \overline{y}\right]} \left\{\left(1 - e^{-\lambda(\hat{y})}\right) y + t_f(\hat{y})\right\} \quad \text{all } y \in \left[\underline{y}, \overline{y}\right], \tag{IC}$$

$$m \int_{\underline{y}}^{\overline{y}} t_f(y)dF(y) + n \cdot t_w = 0, \tag{BB}$$

$$\lambda(y) \equiv (F'(y))^{-1} q(y) b. \tag{15}$$

If the triple $\{q(y), t_f(y), t_w\}$ constitutes a feasible allocation, in the sense that it satisfies the constraints in (15), then the ex-ante expected utilities for the worker and the firm are respectively given by

$$\begin{aligned} E\left[U_w|\omega\right] &= t_w, \\ E\left[U_f|\omega\right] &= \int_{\underline{y}}^{\overline{y}} \left[\left(1 - e^{-\lambda(y)}\right) y + t_f(y)\right] dF(y). \end{aligned} \tag{16}$$

An allocation ω is said to be *constrained Pareto efficient* if it is feasible and is such that there is no other feasible allocation such that both the worker and the firm are weakly better off and one of them is strictly better off, where the welfare criterion is defined by (16).

Notice that utility can be transferred across agents at the constant rate b^{-1} or, more formally, that if $\{q(y), t_f(y), t_w\}$ is feasible then $\left\{q(y), t_f(y) + T, t_w - b^{-1}T\right\}$ is feasible for all $T \in \mathbb{R}$. Moreover, the weighted sum of the agents' expected utility only depends on the total output of the economy, which in turn only depends on $q(y)$. Therefore, ω is constrained Pareto efficient if and only if it is feasible and the associated assignment rule maximizes gross output over the set of feasible allocations.

To begin with, consider the problem of maximizing total output over the set of assignment rules $q(y)$ that satisfy constraint (PM) only. The necessary conditions for the optimality of $q(y)$ are such that: (i) if the workers are assigned with positive probability to both types y_1 and y_2, then the effect on total output of a marginal increase in either $q(y_1)$ or $q(y_2)$ has to be identical, (ii) the worker is never assigned to firms whose productivity is below a threshold y_c. The optimal $q^*(y)$ is such that

$$\frac{b \cdot q^*(y)}{F'(y)} = \lambda^*(y) = \begin{cases} \log\left(\frac{y}{y_c}\right) & \text{if } y > y_c, \\ 0 & \text{if } y \leq y_c, \end{cases} \tag{17}$$

where y_c is the unique solution to the equation

$$\int_{y_c}^{\overline{y}} \log\left(\frac{y}{y_c}\right) dF(y) = b. \tag{18}$$

Next, consider the optimal announcement problem faced by a firm y, given the menu of queues and transfers $\left\{\lambda^*(\hat{y}), t_f^*(\hat{y})\right\}$, where the the function $t_f^*(\hat{y})$ is defined as

$$t_f^*(\hat{y}) = \begin{cases} -y_c \log\left(\frac{\hat{y}}{y_c}\right) & \text{if } \hat{y} > y_c, \\ 0 & \text{if } \hat{y} \leq y_c. \end{cases} \tag{19}$$

The optimal announcement policy $\hat{y}^*(y)$ is a maximizer of the sum of the expected output $y \cdot \left(1 - e^{-\lambda(\hat{y})}\right)$ and the transfer $t_f^*(\hat{y})$ with respect to $\hat{y}$. When $y > y_c$, the maximum of the firm's objective function is strictly positive, the unique maximizer $\hat{y}^*$ is greater than y_c, and at $\hat{y}^*$ the marginal benefit ${\hat{y}^*}^{-2}(y_c y)$ from announcing a higher type is equal to the marginal cost $\hat{y}^{*-1} y_c$. Therefore, for $y > y_c$, the optimal announcement policy for firm y is to truthfully reveal its type. When $y \leq y_c$, the maximum of the firm's objective function is equal to zero and any point in the interval $\left[\underline{y}, y_c\right]$ is a maximizer. Therefore, for $y \leq y_c$, truthful revelation is an optimal announcement policy for firm y.

In conclusion, the assignment rule $q^*(y)$ and the transfer function $t_f^*(y)$ satisfy the constraints (PM) and (IC) and $q^*(y)$ maximizes total output over the set of feasible allocations.

Proposition 6: (Efficient Allocations) *For all $T \in \mathbb{R}$, the set of constrained Pareto efficient allocations contains*

$$\omega^*(T) = \left\{ q^*(y), t_f^*(y) + T, -b^{-1}\left(\int_{\underline{y}}^{\overline{y}} t_f^*(y) dF(y) + T\right)\right\}.$$

Conversely, any constrained Pareto-efficient allocation $\hat{\omega}$ is such that $\hat{q}(y) = q^*(y)$ and such that $(E[U_f|\hat{\omega}], E[U_w|\hat{\omega}])$ is equal to $(E[U_f|\omega^*(T)], E[U_w|\omega^*(T)])$ for some $T \in \mathbb{R}$.

One way to decentralize the family of constrained efficient allocations $\omega^*(T)$ is a mechanism where firms announce and commit to their wage while lump-sum taxes and subsidies transfer utility across agents. On the contrary, the decentralized allocation generated by the the cheap-talk and bargaining games is not constrained efficient (i.e. it fails to maximize gross output) either under directed or random search. There are two externalities to account for such market failure. First, when a worker visits a certain vacancy, she lowers the probability that other applicants will obtain that position and increases the probability that the firm will fill it. But the worker cannot be made to internalize the consequences of her application decision, because the wage is determined as the outcome of a bargaining game. Secondly, when a firm advertises its vacancy with $s(i)$, it affects the informative content of that message and, in turn, the number of workers that it attracts. But the firm cannot internalize this externality, because the property rights on $s(i)$ are not defined.

While neither directed nor random search equilibria are constrained efficient, which one leads to higher welfare depends on the relative strength of the composition and congestion effects of information. More specifically, suppose that in the directed search equilibrium the workers learn the identity of firms whose productivity is above the cutoff $\hat{y} \in \left(\underline{y}, \overline{y}\right)$ before the application stage of the game. On the one hand, this piece of information reduces the number of matches with respect to a random search equilibrium by a measure of

$$CNG = m\left[\left(e^{-b} - e^{-\lambda(1)}\right) F\left(\hat{y}\right) + \left(e^{-b} - e^{-\lambda(2)}\right)\left(1 - F\left(\hat{y}\right)\right)\right],$$

where $(\lambda(1), \lambda(2))$ are uniquely pinned down by the market clearing and worker's optimality conditions (12). On the other hand, the information increases the average productivity of a match with respect to a random search equilibrium by

$$CMP = \frac{\left(1 - e^{-\lambda(1)}\right) F\left(\hat{y}\right)\left(E\left[y|y \leq \hat{y}\right] - E[y]\right) + \left(1 - e^{-\lambda(2)}\right)\left(1 - F\left(\hat{y}\right)\right)\left(E\left[y|y \geq \hat{y}\right] - E[y]\right)}{\left(1 - e^{-\lambda(1)}\right) F\left(\hat{y}\right) + \left(1 - e^{-\lambda(2)}\right)\left(1 - F\left(\hat{y}\right)\right)}.$$

If the difference in average productivity between firms above and below the cutoff $\hat{y}$ is sufficiently small, then information reduces gross output and in turn welfare. Conversely, if the difference between $E\left[y|y \geq \hat{y}\right]$ and $E\left[y|y \leq \hat{y}\right]$ is sufficiently large, then information is welfare improving.

5 Conclusions

In this paper, I have studied a search model of the labor market where the amount of information available to workers about the productivity of different vacancies is determined endogenously as the equilibrium of a communication game where firms advertise their vacancies with messages whose content is neither verifiable nor contractually binding. From the decision theoretic point of view, the firm's trade-off between advertising a low or a high productivity vacancy is that the latter attracts more applicant but it reduces the informational rents extracted at the bargaining stage. From the game theoretic point of view, how the sign of the firm's trade-off varies with the firm's type determines the existence of equilibria where different types have an incentive to post different messages and the informativeness of the cheap-talk. In general equilibrium, when bargaining is fast and the matching process generates a convex labor supply curve, this paper finds that the informativeness of communication depends non-monotonically on the tightness of the labor market. When workers are sufficiently scarce with respect to vacancies, any advertisement for a high productivity vacancy is not credible and the unique equilibrium has firms babbling and workers searching at random. When vacancies are sufficiently scarce with respect to workers, no firm has an incentive to announce its type and the unique equilibrium features random search. When b takes on intermediate values, there also exist an equilibrium where high and low productivity firms separate at the communication stage and search is directed. In the directed search equilibrium, the value of searching, the average wage and the average productivity of a match are greater than in the random search equilibrium, but the job-finding rate is lower. Therefore information creates both a positive composition effect and a negative congestion effect and, depending on the relative strength of the two, directed search might or might not be welfare improving over random search.

There are many real-world markets where potential trading partners have private information about characteristics that influence the gains from trade, they can (almost) costlessly communicate with one another before any relationship-specific investment are made and the truthfulness of the communication is neither legally enforceable nor immediately verifiable. In the labor market, worker have superior information about their skills and firms have superior information about which type and how much labor they need, many job search web-sites are free, but verifying a resumé is a costly activity. In the online dating market, people post their profile and search large databases free of charge, but the only way to verify that the profile meets the real characteristics of a person is to go on a time-consuming date. In the mobile phone service market, comparing the price advertised by different provider is almost effortless but such price does not always reflect the buyer's cost. These observations suggest that this paper can be fruitfully extended to allow for two sided heterogeneity, multiple privately observed attributes, repeat purchases and more general price setting mechanisms to better understand the relative importance of communication and search in matching agents on one side of the market to those on the other side.

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A Appendix

Proof of Proposition 1: The proof is divided in four steps. First, some standard properties of any sequential equilibrium are derived. Then, the wage at which a firm with productivity y trades is bounded using condition 3. In the third step, I verify that the assumptions of theorem 1 in Gul and Sonnenschein (1988) are verified by any sequential equilibrium that satisfies conditions 1 and 2. Finally, I combine the results to derive the limiting outcome of the bargaining game.

Claim 1*: Consider any sequential equilibrium of the bargaining game starting with a prior belief $g_{j,i}\left(h^2\right) = G$, $G'(y) > 0$ iff $y \in [l,h]$. Suppose that the game has reached period $t \geq 2$ and generated history h^t and the worker's beliefs are given by some $g_{j,i}\left(h^t\right)$ with support on $[b,a]$, $l \leq b \leq a \leq h$, then:*

(i) the wage at which type y trades is non-increasing in y and the period in which type y trades is non-decreasing in y;

(ii) every type $y \in [b,a]$ accepts the wage demand w if $w < \rho \cdot b$, and rejects it if $w > \rho \cdot a$;

(iii) the worker rejects the wage offer w if w is smaller than $\exp(-(1-\beta)\Delta)(\rho \cdot b)$*, and accepts it if w is greater than* $\exp(-(1-\beta)\Delta)(\rho \cdot a)$.

Proof: (i) Denote with w_j and τ_j the wage and the time at which type y_j trades, $b \leq y_1 \leq y_2 \leq a$, conditional on having reached h^t. By revealed preferences, the following inequalities have to hold

$$e^{-\tau_j}\left(y_j - w_j\right) \geq e^{-\tau_{-j}}\left(y_j - w_{-j}\right), \text{ for } j = 1,2, \tag{20}$$

and therefore $w_1 \leq w_2$, $\tau_1 \geq \tau_2$. (ii)–(iii) Denote with $\overline{w}$ the supremum of the wages that the worker gets in any equilibrium and after any history that is a subsequent of h^t. Therefore, any type $y \in [b,a]$ can obtain the payoff $y - \exp\left(-(1-\beta)\Delta\right) \cdot \overline{w}$ whenever it is its turn to move by making the wage offer $\exp\left(-(1-\beta)\Delta\right) \cdot \overline{w}$. Moreover, every type $y \in [b,a]$ will reject a wage demand w if

$$w > a - \exp\left(-\beta\Delta\right) \cdot \left(a - \exp\left(-(1-\beta)\Delta\right) \cdot \overline{w}\right). \tag{21}$$

Combining these two results, it is immediate to conclude that the supremum $\overline{w}$ is smaller or equal than $\rho \cdot a$. Following a similar reasoning, it can be shown that the infimum $\underline{w}$ of the wages is greater or equal than $\exp(-(1-\beta)\Delta)(\rho \cdot b)$. Q.E.D.

Claim 2: *If the sequential equilibrium of the bargaining game satisfies conditions 1 and 3, then for any $y \in [l,h]$ the wage $w(y)$ at which type y trades belongs to the interval*

$$\left[e^{-(1-\beta)\Delta}\left(\frac{1-e^{-\beta\Delta}}{1-e^{-\Delta}}\right) \cdot l, \left(1 - e^{-\Delta} + e^{-(2-\beta)\Delta}\left(\frac{1-e^{-\beta\Delta}}{1-e^{-\Delta}}\right)\right) \cdot y\right]. \tag{22}$$

PROOF: Suppose that the equilibrium prescribes that in period t, the worker demands a wage w_t which is accepted by firm y_1. In light of claim 1, w_t is also accepted by types in the interval $[y_t, a]$ with $l \leq y_t \leq y_1$. If $y_t = l$, then condition 3 implies that if w_t is rejected then the worker's posterior belief is degenerate at l and therefore she will accept any counteroffer greater or equal than $\exp(-(1-\beta)\Delta)(\rho \cdot y_t)$. If $y_t > l$, then there exists an equilibrium counteroffer w_{t+1} that is advanced by types $y \in [y_{t+1}, y_t]$. If $y_{t+1} = l$, then condition 3 implies that w_{t+1} is smaller or equal to $\exp(-(1-\beta)\Delta)(\rho \cdot y_t)$. If $y_{t+1} > l$, then there exists an equilibrium unacceptable counter-offer $\hat{w}_{t+1} < w_{t+1}$ from types $y \in [l, y_{t+1}]$. Therefore, the firm can reject w_{t+2} and trade in period $t+3$ by offering any wage w_{t+3} greater or equal than $\exp(-(1-\beta)\Delta)(\rho \cdot y_{t+1})$. Overall, using revealed preferences, if firm y trades for $w(y)$ then $w(y)$ is smaller or equal than the right hand side of (22). The left hand side of (22) is derived from claim 1. Q.E.D.

CLAIM 3: (Gul and Sonnenschein, 1988) *Take any sequential equilibrium of the bargaining game that satisfies conditions 1–2, then for any $\epsilon > 0$, there exists a $\bar{\Delta}(\epsilon) > 0$, such that for all $\Delta < \bar{\Delta}(\epsilon)$ the probability that trade occurs in ϵ time is at least $1 - \epsilon$.*

PROOF: Conditions 1–2 are precisely the Stationarity of Buyer's Strategies and the Monotonicity requirements in Gul and Sonnenschein (1988). Moreover, in light of lemma 3.3 in Grossman and Perry (1986b), without loss of generality it can be assumed that the No-Free Screening requirement is satisfied. With minimal adaptations, it is possible to apply theorem 1 in Gul and Sonnenschein.

Denote with $\{\Delta(n)\}_{n=1}^{\infty}$ a positive decreasing sequence such that $\lim_{n\to\infty}\Delta(n) = 0$. Let $\epsilon(1) > 0$ be such that $\Delta(1) < \bar{\Delta}(\epsilon(1))$ and set, recursively, $\epsilon(n+1)$ to $\epsilon(n)/2$ if $\Delta(n+1) < \bar{\Delta}(\epsilon(n)/2)$ and to $\epsilon(n)$ otherwise. The sequence $\{\epsilon(n)\}_{n=1}^{\infty}$ is decreasing and converges to zero, because of claim 3. For $n = 1, 2, \ldots$, let $\theta(n)$ be a sequential equilibrium from $\Theta(\Delta(n); G, \beta)$. Given the equilibrium $\theta(n)$, denote with y_n^* the lowest type of firm for which $\tau_n(y_n^*) < \epsilon(n)$ and with $w_n(y_n^*)$ the wage at which it trades. From individual rationality, we know that

$$(y - w_n(y))\, e^{-\tau_n(y)} \geq (y - w_n(y_n^*))\, e^{-\tau_n(y_n^*)} \qquad \text{all } y \in [l, h] \tag{23}$$

An upper bound on the firm's expected payoff is obtained from claim 1

$$(y - w_n(y))\, e^{-\tau_n(y)} \leq (y - \exp(-\Delta(n))\, \rho_n \cdot l) \qquad \text{all } y \in [l, h] \tag{24}$$

The sequence $\{y_n^*\}_{n=1}^{\infty}$ converges to l, because of claim 3. The sequence $\{w_n(y_n^*)\}_{n=1}^{\infty}$ converges to $\beta \cdot l$ because of (22) and $\beta \cdot l$ is also the limit of $\exp(-\Delta(n))\, r_n$. Therefore, the sequence $\left\{(y - w_n(y))\, e^{-\tau_n(y)}\right\}_{n=1}^{\infty}$ converges to $y - \beta \cdot l$ for all y in $[l, h]$. Q.E.D.

Proof of Lemma 1: (i) Given the normalization on the ordering of messages and the wage determination equation (**??**), it follows that $w^*(1)$ equals $\beta \underline{y}$. By revealed preferences, the expected profit of a firm y is bounded below by $P^*(1)(y - \beta \underline{y})$. If the probability $P^*(1)$ of filling a vacancy that pays $\beta \underline{y}$ is positive, then every firm y must obtain positive profits by posting a message $s(i)$ such that $p^*(k|y) > 0$. Next, consider the case where there is an integer n greater than 1 and such that $P^*(1) = ...P^*(n-1) = 0$ and $P^*(n) > 0$. Every firm with productivity y greater than $\beta \underline{y}^*(n)$ makes positive profits, because $P^*(n)\left(y - \beta \underline{y}^*(n)\right)$ is positive, and follows an advertisement strategy such that $\sum_{j=n}^{N} p^*\left(i|y\right) = 1$. Because the measure of firms in the open interval $\left(\beta \underline{y}^*(n), \underline{y}^*(n)\right)$ is positive, there is a $j \geq n$ such that the lower bound on $G^*\left(y|j\right)$ is smaller than $\underline{y}^*(n)$. The wage for such $s(l)$ is $w^*\left(l\right) < w^*(n)$ which contradicts the assumption on the ordering of messages. (ii) From the previous argument, it follows that $P^*(k) > 0$ for all $k = 1, 2, ...N$. Q.E.D.

Proof of Lemma 2: Suppose that type's y expected payoff from posting the message $s(k)$ is given by $P^*\left(k\right) \cdot \left(y - w(k)\right)$ for any integer $k \in [1, N]$. Given these payoff function, type y prefers $s(k)$ to $s(l)$ if and only if

$$(P^*(k) - P^*(l)) \cdot y_1 \geq P^*(k)w^*(k) - P^*(l)w^*(l). \tag{25}$$

From (25), it follows that if y_1 prefers $s(k)$ over $s(l)$ and $k > l$, then every type y greater than y_1 also prefers $s(k)$. Similarly, if y_1 prefers $s(l)$ over $s(k)$ and $k > l$, then every type y smaller than y_1 also prefers $s(l)$.

Next, suppose that there exists an integer $k \in [1, N-1]$ such that $\underline{y}^*\left(k+1\right)$ is not indifferent between $s(k)$ and $s(k+1)$. This implies that

$$P^*(k)\left(\underline{y}^*\left(k+1\right) - w^*(k)\right) \neq P^*(k+1)\left(\underline{y}^*\left(k+1\right) - w^*(k+1)\right). \tag{26}$$

If the left hand side of (26) is greater than the right hand side, then there exists an $\epsilon_1 > 0$ such that every type y smaller than $\underline{y}^*\left(k+1\right) + \epsilon_1$ strictly prefers $s(k)$ to $s(k+1)$. Therefore, the infimum y over $\left[\underline{y}, \overline{y}\right]$ such that $p^*(k|y) > 0$ is greater than $\underline{y}^*\left(k+1\right)$. If the left hand side of (26) is smaller than the right hand side, then there exists an $\epsilon_2 > 0$ such that every type y greater than $\underline{y}^*\left(k+1\right) - \epsilon_2$ strictly prefers $s(k+1)$ to $s(k)$. Therefore, the infimum y over $\left[\underline{y}, \overline{y}\right]$ such that $p^*(k|y) > 0$ is smaller than $\underline{y}^*\left(k+1\right)$.

If the payoff function was $P^*\left(k\right) \cdot \left(y - w(k)\right)$, then in any N-message equilibrium the optimal advertising strategy would be $p^*\left(k|y\right) = 1$ for all y in the interval $\left(\underline{y}^*\left(k\right), \underline{y}^*\left(k+1\right)\right)$ and the type $\underline{y}^*\left(k+1\right)$ would be indifferent between $s(k)$ and $s(k+1)$. But given lemma 1, the optimality condition (5) is satisfied if and only if the message $s(k)$ maximizes the payoff function $P^*\left(l\right) \cdot \left(y - w(l)\right)$ for $1 \leq l \leq N$. Q.E.D.

PROOF OF LEMMA 3: (i) Take an arbitrary $x(1)$ in $\mathbb{R}_{++}$ and define $\psi(z)$ as

$$\psi(z) = \frac{z}{U}\Psi(z, x(1); U) = (1-\beta) + \beta\frac{x(1)}{z} - \frac{(1-\exp(-x(1)))}{(1-\exp(-z))}.$$

For every $z \in (x, \infty)$, $\psi(z)$ takes the same sign as $\Psi(z, x; U)$. Therefore the set $Z^*(x(1))$ coincides with the set of solutions of the equation $\psi(z) = 0$, which is independent from U.

(ii, iii, iv) The function $\psi(z)$ is twice continuously differentiable and at the boundaries of the domain converges to

$$\begin{aligned} &\lim_{z\to x(1)} \psi(z) = 0, \\ &\lim_{z\to\infty} \psi(z) = \exp(-x(1)) - \beta. \end{aligned} \tag{27}$$

The first derivative of $\psi(z)$ is given by

$$\psi'(z) = \exp(-z)\frac{(1-\exp(-x(1)))}{(1-\exp(-z))^2} - \beta\frac{x(1)}{z^2}, \tag{28}$$

and –at the boundaries– it converges to

$$\begin{aligned} &\lim_{z\to x(1)} \psi'(z) = \frac{\exp(-x(1))}{(1-\exp(-x(1)))} - \beta\frac{1}{x(1)}, \\ &\lim_{z\to\infty} \psi'(z) = 0. \end{aligned} \tag{29}$$

Finally notice that $z^2\psi'(z)$ is a linear transformation of the derivative of $(1-\exp(-z))^{-1}z$. Since $(1-\exp(-z))^{-1}z$ is a strictly convex function, it follows that the equation $\psi'(z)z^2 = 0$ has at most one solution and so does $\psi'(z) = 0$.

If $x(1) < \log(1/\beta) < h^{-1}(\beta)$, then the function $\psi(z)$ is strictly positive for $z \to \infty$ and its derivative is positive for $z \to x(1)$. Since there is at most one maximum/minimum, it follows that $\psi(z) > 0$ for all $z \in (x(1), \infty)$. If $x(1) > h^{-1}(\beta) > \log(1/\beta)$, the function $\psi(z)$ is strictly negative for $z \to \infty$ and its derivative is negative for $z \to x(1)$. Therefore $\psi(z) < 0$ for all $z \in (x(1), \infty)$. When $h^{-1}(\beta) > x(1) > \log(1/\beta)$, the function $\psi(z)$ is strictly negative for $z \to \infty$ and its derivative is positive for $z \to x(1)$. There is a unique global maximum and a unique zero $z^*(x(1))$. For $z < (>) z^*(x(1))$, $\psi(z)$ is positive (*negative*).

(v) Take any $(x(1), z(1))$ such that $x(1)$ belongs to $(\log(1/\beta), h^{-1}(\beta))$ and $z(1) > x(1)$, then the derivative of $\psi(z; x)$ with respect to x is

$$\frac{\partial\psi(z;x)}{\partial x}\Big|_{x=x(1)} = \frac{\beta}{z(1)} - \frac{\exp(-x(1))}{(1-\exp(-z(1)))}. \tag{30}$$

By assumption β belongs to $(\exp(-x(1)), h(x(1)))$ and therefore

$$\frac{\partial \psi\left(z;x\right)}{\partial x}|_{x=x(1)} < \frac{\exp(-x(1))}{(1-\exp(-x(1)))}\frac{x(1)}{z(1)} - \frac{\exp(-x(1))}{(1-\exp(-z(1)))} < 0. \tag{31}$$

The first implication of inequality (31) is that the function $z^{*}(x)$ is continuous. Secondly, (31) implies that if $x(1)$ and $x(2)$ are in $(\log(1/\beta), h^{-1}(\beta))$ and $x(1) < x(2)$, then the intersection $z^{*}(x(2))$ between $\psi\left(z; x(2)\right)$ and the horizontal axis is strictly smaller than $z^{*}(x(1))$. In turn, the continuity and monotonicity of the solution to $\psi\left(z;x\right) = 0$ imply that $\lim_{x\to\log(1/\beta)} z(x) = \infty$ and $\lim_{x\to h^{-1}(\beta)} z(x) = h^{-1}(\beta)$. Q.E.D.

Proof of Theorem 1: (i) It is immediate to verify that the unique solution to the system (N&S) for $N = 1$ is given by $\left(\lambda(1), \underline{y}\left(1\right)\right) = \left(b, \underline{y}\right)$. (ii) The vector $\left(\lambda(1), \lambda(2), \underline{y}\left(1\right), \underline{y}\left(2\right)\right)$ is a solution to $(N\&S)$ for $N = 2$ if and only if $\lambda(2)$ is equal to $z^{*}\left(\lambda(1)\right)$, $\underline{y}\left(1\right)$ is equal to $\underline{y}$, $\underline{y}\left(2\right) = \underline{y}_{2}(\lambda(1)) \leq \overline{y}$ and the queue length $\lambda(1)$ lies in X and solves the equation

$$\Gamma(\lambda(1)) \equiv \lambda(1) + (z^{*}(\lambda(1)) - \lambda(1))(1 - F(\underline{y}_{2}(\lambda(1)))) = b, \tag{32}$$

where

$$\underline{y}_{2}(\lambda(1)) \equiv \underline{y}\frac{z^{*}\left(\lambda(1)\right)}{\lambda(1)}\frac{(1-\exp(-\lambda(1)))}{(1-\exp(-z^{*}(\lambda(1))))}.$$

From the properties of $z^{*}(x)$ derived in lemma 3, it follows that: (i) the function $\Gamma : X \rightarrow \mathbb{R}_{+}$ is continuous, (ii) it is bounded below by the 45-degree line, (iii) it takes the value $\lambda(1)$ for $\lambda(1) \in (\log\left(1/\beta\right), \underline{\Gamma}]$ where $\underline{\Gamma}$ is the unique solution to $\underline{y}_{2}(x) = \overline{y}$, (iv) it is bounded above by z* $\left(\underline{x}\right)$, (v) the maximum $\overline{\Gamma}$ of $\Gamma(\lambda(1))$ exists and is greater or equal than $h^{-1}(\lambda(1))$. If b is smaller than $\log\left(1/\beta\right)$, then $\Gamma(\lambda(1)) > b$ for all $\lambda(1) \in X$. If b belongs to the interval $(\log\left(1/\beta\right), \underline{\Gamma})$, then the unique solution to (16) is $\lambda(1) = b$ and $\underline{y}_{2}(b) > \overline{y}$. If b lies in the interval $\left[\underline{\Gamma}, \overline{\Gamma}\right]$, then there exists at least one solution to (32) such that $\lambda(1) \in X$ and $\underline{y}_{2}(\lambda(1)) \leq \overline{y}$. Finally, if b is greater than $\overline{\Gamma}$, the function $\Gamma(\lambda(1)) < b$ for all $\lambda(1) \in X$. Q.E.D.

Proof of Proposition 5: (i) From the strict concavity of $(1 - e^{-\lambda})$ with respect to λ, it follows that for any $\alpha \in (0, 1)$ and any $\lambda(1)$, $\lambda(2)$

$$\alpha\left(1 - e^{-\lambda(1)}\right) + (1-\alpha)\left(1 - e^{-\lambda(2)}\right) < \left(1 - e^{-\alpha\lambda(1)-(1-\alpha)\lambda(2)}\right). \tag{33}$$

In particular let $\alpha = F\left(\underline{y}^{*}(2)\right)$, $\lambda(1) = \lambda^{*}(1)$ and $\lambda(2) = \lambda^{*}(2)$, then inequality (33) implies $AJF_{r} > AJF_{d}$. (ii-iii) These properties are immediately derived from the definitions of ALP_{r}, ALP_{d}, AW_{r} and AW_{d}. Q.E.D.

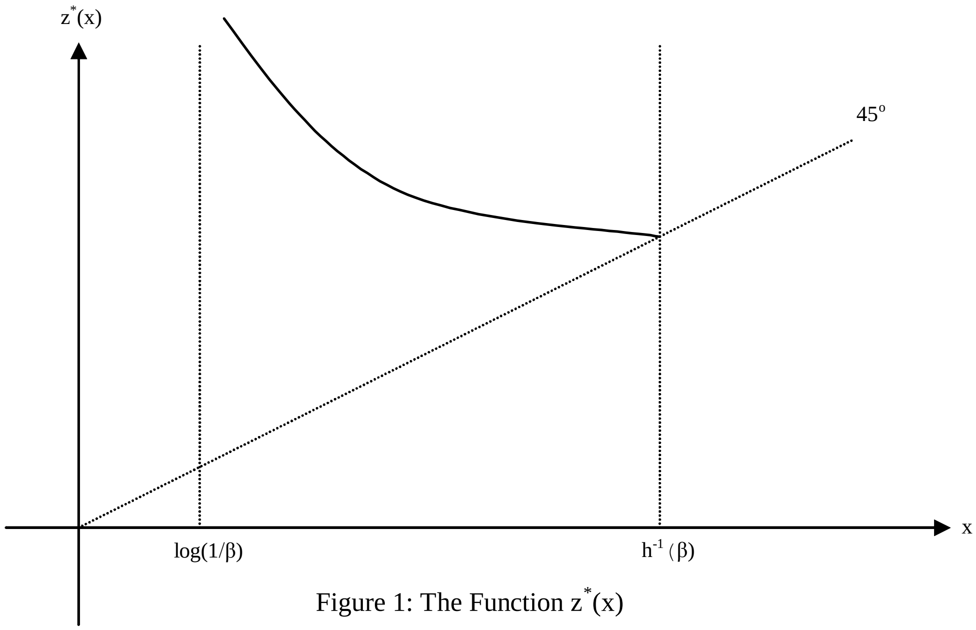

Figure 1: The Function $z^*(x)$

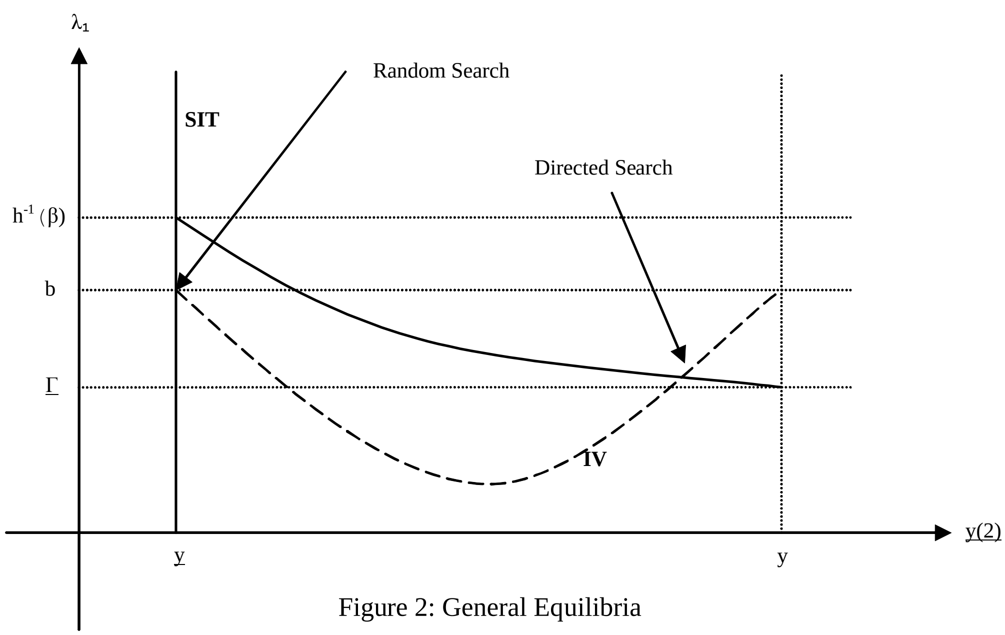

Figure 2: General Equilibria